Two finiteness theorem for (a, b)-modules

Daniel Barlet

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16/01/08

Summary.

We prove the following two results

1. For a proper holomorphic function $f : X \to D$ of a complex manifold X on a disc such that $\{df = 0\} \subset f^{-1}(0)$, we construct, in a functorial way, for each integer p, a geometric (a,b)-module E^p associated to the (filtered) Gauss-Manin connexion of f.
This first theorem is an existence/finiteness result which shows that geometric (a,b)-modules may be used in global situations.

2. For any regular (a,b)-module E we give an integer $N(E)$, explicitely given from simple invariants of E, such that the isomorphism class of $E/b^{N(E)}.E$ determines the isomorphism class of E.
This second result allows to cut asymptotic expansions (in powers of b) of elements of E without loosing any information.

AMS Classification : 32-S-05, 32S 20, 32-S-25, 32-S-40.

Key words : (a,b)-module or Brieskorn modules, Gauss-Manin connexion, vanishing cycles.

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Contents

1 Introduction.

The following situation is frequently met : we consider a vector space E of multi-valued holomorphic functions (possibly with values in a complex finite dimensional vector space V) with finite determination on a punctured disc around 0 in $\mathbb{C}$, stable by multiplication by the variable z and by "primitive". These functions are determined by there formal asymptotic expansions at 0 of the type

$$\sum_{(\alpha,j)\in A\times[0,n]} c_{\alpha,j}(z).z^{\alpha}.\frac{(Logz)^j}{j!}$$

where n is a fixed integer, where A is a finite set of complex numbers whose real parts are, for instance, in the interval $]-1,0]$, and where the $c_{\alpha,j}$ are in $\mathbb{C}[[z]]\otimes V$. Define $e(\alpha,j) = z^{\alpha}.(Logz)^j/j!$ and $E(\alpha,n) := \oplus_{j=0}^{n}\ (\mathbb{C}[[z]]\otimes V).e(\alpha,j)$. Then for each α, $E(\alpha,n)$ is a free $\mathbb{C}[[z]]-$module of rank $(n+1).\dim V$ which is stable by $b := \int_0^z$. To be precise, b is defined by induction on $j \geq 0$ by the "obvious" formulas :

$$b[e(\alpha,0)] = \frac{e(\alpha+1,0)}{\alpha+1} \quad \text{and for} \quad j \geq 1$$

$$b[e(\alpha,j)] = \frac{e(\alpha+1,j)}{\alpha+1} - \frac{1}{\alpha+1}.b[e(\alpha,j-1)].$$

So we have an inclusion $E \subset \oplus_{\alpha\in A}E(\alpha)$ which is compatible with $a := \times z$ and by b. Note that each $E(\alpha)$ is also a free finite rank $\mathbb{C}[[b]]-$module which is stable by a.

Let us assume now that E is also a $\mathbb{C}[[b]]-$module which is stable by a. Then E has to be free and of finite rank over $\mathbb{C}[[b]]$. Our aim in this situation is to understand and to describe the relations between the coefficients $c_{\alpha,j}$ of the asymptotic expansions of elements in E. This leads to construct "invariants" associated to the given E.

As in the "geometric" situations we consider the complex numbers $\alpha \in A$ are rationnal numbers, a change of variable of type $t := z^{1/N}$, with $N \in \mathbb{N}^*$, allows to reduce the situation to the case where all α are 0. Then one can use the "nilpotent" operator $\frac{d}{dt}$ and the nilpotent part of the monodromy $e_{\alpha,j} \to e_{\alpha,j-1}$ to construct some filtrations in order, in the good cases, to build a Mixte Hodge structure on a finite dimensional vector space associated to E. For instance, this is the case in A.N. Varchenkho's description[V. 80] of the Mixte Hodge structure built by J. Steenbrink [St. 76] on the cohomology of the Milnor fiber of an holomorphic function with an isolated singularity at the origin of $\mathbb{C}^{n+1}$.

But it is clear that we loose some information in this procedure. The point of view which is to consider E itself as a left module on the (non commutative) algebra

$$\tilde{\mathcal{A}} := \{\sum_{\nu\geq 0} P_{\nu}(a).b^{\nu}\}$$

where the P_ν are polynomials with complex coefficients is richer. This is evidenced by M. Saito result [Sa. 91].

The aim of the first part of this article is to build in a natural way, for any proper holomorphic function $f : X \to D$ of a complex manifold X, assumed to be smooth outside its $0-$fiber $X_0 := f^{-1}(0)$, a regular (geometric) (a,b)-module for each degree $p \geq 0$, which represent a filtered version of the Gauss-Manin connexion of f at the origin.
This result is in fact a finiteness theorem which is a first step to refine the limite Mixte Hodge structure in this situation. It is interesting to remark that no Kähler assumption is used in this construction of these geometric (a,b)-modules.
This obviuosly shows that (a,b)-modules are basic objects and that they are important not only in the study of local singularities of holomorphic functions but more generally in complex geometry . So it is interesting to have some tools in order to compute them.
This is precisely the aim of the second part of this paper. We prove a finiteness result which gives, for a regular (a,b)-module E, an integer $N(E)$, bounded by simple numerical invariants of E, such that you may cut the asymptotic expansions (in powers of b) of elements of E without any lost of information on the structure of the (a,b)-module E. It is well known that the formal asymptotic expansions for solutions of a regular differential system always converge, and also that such an integer exists for any meromorphic connexion in one variable (see [M.91] proposition 1.12). But it is important to have an effective bound for such an integer easely computable from simple invariants of the (a,b)-module structure of E.

2 The existence theorem.

2.1 Preliminaries.

Here we shall complete and precise the results of the section 2 of [B.07]. The situation we shall consider is the following : let X be a connected complex manifold of dimension $n+1$ and $f : X \to \mathbb{C}$ a non constant holomorphic function such that $\{x \in X / \ df = 0\} \subset f^{-1}(0)$. We introduce the following complexes of sheaves supported by $X_0 := f^{-1}(0)$

1. The formal completion "in f" $(\hat{\Omega}^\bullet, d^\bullet)$ of the usual holomorphic de Rham complex of X.

2. The sub-complexes $(\hat{K}^\bullet, d^\bullet)$ and $(\hat{I}^\bullet, d^\bullet)$ of $(\hat{\Omega}^\bullet, d^\bullet)$ where the subsheaves $\hat{K}^p$ and $\hat{I}^{p+1}$ are defined for each $p \in \mathbb{N}$ respectively as the kernel and the image of the map
$$\wedge df : \hat{\Omega}^p \to \hat{\Omega}^{p+1}$$
given par exterior multiplication by df. We have the exact sequence
$$0 \to (\hat{K}^\bullet, d^\bullet) \to (\hat{\Omega}^\bullet, d^\bullet) \to (\hat{I}^\bullet, d^\bullet)[+1] \to 0. \tag{1}$$

Note that $\hat{K}^0$ and $\hat{I}^0$ are zero by definition.

3. The natural inclusions $\hat{I}^p \subset \hat{K}^p$ for all $p \geq 0$ are compatible with the differential d. This leads to an exact sequence of complexes
$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0. \tag{2}$$

4. We have a natural inclusion $f^*(\hat{\Omega}^1_{\mathbb{C}}) \subset \hat{K}^1 \cap Ker\, d$, and this gives a subcomplex (with zero differential) of $(\hat{K}^\bullet, d^\bullet)$. As in [B.07], we shall consider also the complex $(\tilde{K}^\bullet, d^\bullet)$ quotient. So we have the exact sequence
$$0 \to f^*(\hat{\Omega}^1_{\mathbb{C}}) \to (\hat{K}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to 0. \tag{3}$$

   We do not make the assumption here that $f = 0$ is a reduced equation of X_0, and we do not assume that $n \geq 2$, so the cohomology sheaf in degree 1 of the complex $(\hat{K}^\bullet, d^\bullet)$, which is equal to $\hat{K}^1 \cap Ker\, d$ does not coïncide, in general, with $f^*(\hat{\Omega}^1_{\mathbb{C}})$. So the complex $(\tilde{K}^\bullet, d^\bullet)$ may have a non zero cohomology sheaf in degree 1.

Recall now that we have on the cohomology sheaves of the following complexes $(\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}^1_{\mathbb{C}}), (\tilde{K}^\bullet, d^\bullet)$ natural operations a and b with the relation $a.b - b.a = b^2$. They are defined in a naïve way by
$$a := \times f \quad \text{and} \quad b := \wedge df \circ d^{-1}.$$
The definition of a makes sens obviously. Let me precise the definition of b first in the case of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ with $p \geq 2$: if $x \in \hat{K}^p \cap Ker\, d$ write $x = d\xi$ with $\xi \in \hat{\Omega}^{p-1}$ and let $b[x] := [df \wedge \xi]$. The reader will check easily that this makes sens. For $p = 1$ we shall choose $\xi \in \hat{\Omega}^0$ such that $\xi = 0$ on the smooth part of X_0 (set theoretically). This is possible because the condition $df \wedge d\xi = 0$ allows such a choice : near a smooth point of X_0 we can choose coordinnates such $f = x_0^k$ and the condition on ξ means independance of $x_1, \cdots, x_n$. Then ξ has to be (set theoretically) locally constant on X_0 which is locally connected. So we may kill the value of such a ξ along X_0.
The case of the complex $(\hat{I}^\bullet, d^\bullet)$ will be reduced to the previous one using the next lemma.

Lemme 2.1.1 *For each $p \geq 0$ there is a natural injective map*
$$\tilde{b} : \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$$
which satisfies the relation $a.\tilde{b} = \tilde{b}.(a + b)$. For $p \neq 1$ this map is bijective.

PROOF. Let $x \in \hat{K}^p \cap Ker\, d$ and write $x = d\xi$ where $\xi \in \hat{\Omega}^{p-1}$ (with $\xi = 0$ on X_0 if $p = 1$), and set $\tilde{b}([x]) := [df \wedge \xi] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. This is independant on the choice of ξ because, for $p \geq 2$, adding $d\eta$ to ξ does not modify the result as $[df \wedge d\eta] = 0$. For $p = 1$ remark that our choice of ξ is unique.
This is also independant of the the choice of x in $[x] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ because adding $\theta \in \hat{K}^{p-1}$ to ξ does not change $df \wedge \xi$.
Assume $\tilde{b}([x]) = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$; this means that we may find $\alpha \in \hat{\Omega}^{p-2}$ such $df \wedge \xi = df \wedge d\alpha$. But then, $\xi - d\alpha$ lies in $\hat{K}^{p-1}$ and $x = d(\xi - d\alpha)$ shows that $[x] = 0$. So $\tilde{b}$ is injective.
Assume now $p \geq 2$. If $df \wedge \eta$ is in $\hat{I}^p \cap Ker\, d$, then $df \wedge d\eta = 0$ and $y := d\eta$ lies in $\hat{K}^p \cap Ker\, d$ and defines a class $[y] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ whose image by $\tilde{b}$ is $[df \wedge \eta]$. This shows the surjectivity of $\tilde{b}$ for $p \geq 2$.
For $p = 1$ the map $\tilde{b}$ is not surjective (see the remark below).
To finish the proof let us to compute $\tilde{b}(a[x] + b[x])$. Writing again $x = d\xi$, we get

$$a[x] + b[x] = [f.d\xi + df \wedge \xi] = [d(f.\xi)]$$

and so

$$\tilde{b}(a[x] + b[x]) = [df \wedge f.\xi] = a.\tilde{b}([x])$$

which concludes the proof. ∎

Denote by $i : (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet)$ the natural inclusion and define the action of b on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ by $b := \tilde{b} \circ \mathcal{H}^p(i)$. As i is $a-$linear, we deduce the relation $a.b - b.a = b^2$ on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ from the relation of the previous lemma.

The action of a on the complex $([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ is obvious and the action of b is zero.

The action of a and b on $f^*(\hat{\Omega}^1_{\mathbb{C}}) \simeq E_1 \otimes \mathbb{C}_{X_0}$ are the obvious one, where E_1 is the rank 1 (a,b)-module with generator e_1 satisfying $a.e_1 = b.e_1$ (or, if you prefer, $E_1 := \mathbb{C}[[z]]$ with $a := \times z$, $b := \int_0^z$ and $e_1 := 1$).
Remark that the natural inclusion $f^*(\hat{\Omega}^1_{\mathbb{C}}) \hookrightarrow (\hat{K}^\bullet, d^\bullet)$ is compatible with the actions of a and b. The actions of a and b on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$ are simply induced by the corresponding actions on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$.

REMARK. The exact sequence of complexes (1) induces for any $p \geq 2$ a bijection

$$\partial^p : \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$$

and a short exact sequence

$$0 \to \mathbb{C}_{X_0} \to \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \xrightarrow{\partial^1} \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \to 0 \qquad (@)$$

because of the de Rham lemma. Let us check that for $p \geq 2$ we have $\partial^p = (\tilde{b})^{-1}$ and that for $p = 1$ we have $\partial^1 \circ \tilde{b} = Id$. If $x = d\xi \in \hat{K}^p \cap Ker\, d$ then $\tilde{b}([x]) = [df \wedge \xi]$ and $\partial^p[df \wedge \xi] = [d\xi]$. So $\partial^p \circ \tilde{b} = Id \quad \forall p \geq 0$. For $p \geq 2$ and

$df \wedge \alpha \in \hat{I}^p \cap Ker\, d$ we have $\partial^p[df \wedge \alpha] = [d\alpha]$ and $\tilde{b}[d\alpha] = [df \wedge \alpha]$, so $\tilde{b} \circ \partial^p = Id$. For $p = 1$ we have $\tilde{b}[d\alpha] = [df \wedge (\alpha - \alpha_0)]$ where $\alpha_0 \in \mathbb{C}$ is such that $\alpha_{|X_0} = \alpha_0$. This shows that in degree 1 $\tilde{b}$ gives a canonical splitting of the exact sequence (@).

2.2 $\tilde{\mathcal{A}}-$structures.

Let us consider now the $\mathbb{C}-$algebra

$$\tilde{\mathcal{A}} := \{\sum_{\nu \geq 0} P_\nu(a).b^\nu\}$$

where $P_\nu \in \mathbb{C}[z]$, and the commutation relation $a.b - b.a = b^2$, assuming that left and right multiplications by a are continuous for the $b-$adic topology of $\tilde{\mathcal{A}}$. Define the following complexes of sheaves of left $\tilde{\mathcal{A}}-$modules on X :

$$(\Omega'^{\bullet}[[b]], D^{\bullet}) \quad \text{and} \quad (\Omega''^{\bullet}[[b]], D^{\bullet}) \quad \text{where} \tag{4}$$

$$\Omega'^{\bullet}[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{K}^p$$

$$\Omega''^{\bullet}[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{I}^p$$

$$D(\sum_{j=0}^{+\infty} b^j.\omega_j) = \sum_{j=0}^{+\infty} b^j.(d\omega_j - df \wedge \omega_{j+1})$$

$$a.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=0}^{+\infty} b^j.(f.\omega_j + (j-1).\omega_{j-1}) \quad \text{with the convention} \quad \omega_{-1} = 0$$

$$b.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=1}^{+\infty} b^j.\omega_{j-1}$$

It is easy to check that D is $\tilde{\mathcal{A}}-$linear and that $D^2 = 0$. We have a natural inclusion of complexes of left $\tilde{\mathcal{A}}-$modules

$$\tilde{i} : (\Omega''^{\bullet}[[b]], D^{\bullet}) \to (\Omega'^{\bullet}[[b]], D^{\bullet}).$$

Remark that we have natural morphisms of complexes

$$u : (\hat{I}^{\bullet}, d^{\bullet}) \to (\Omega''^{\bullet}[[b]], D^{\bullet})$$
$$v : (\hat{K}^{\bullet}, d^{\bullet}) \to (\Omega'^{\bullet}[[b]], D^{\bullet})$$

and that these morphisms are compatible with i. More precisely, this means that we have the commutative diagram of complexes

$$\begin{array}{ccc} (\hat{I}^{\bullet}, d^{\bullet}) & \xrightarrow{u} & (\Omega''^{\bullet}[[b]], D^{\bullet}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{i}} \\ (\hat{K}^{\bullet}, d^{\bullet}) & \xrightarrow{v} & (\Omega'^{\bullet}[[b]], D^{\bullet}) \end{array}$$

The following theorem is a variant of theorem 2.2.1. of [B. 07].

Théorème 2.2.1 *Let X be a connected complex manifold of dimension $n+1$ and $f : X \to \mathbb{C}$ a non constant holomorphic function with the following condition:*

$$\{x \in X/\ df = 0\} \subset f^{-1}(0).$$

Then the morphisms of complexes u and v introduced above are quasi-isomorphisms. Moreover, the isomorphims that they induce on the cohomology sheaves of these complexes are compatible with the actions of a and b.

This theorem builds a natural structure of left $\tilde{\mathcal{A}}-$modules on each of the complex $(\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}^1_{\mathbb{C}}), (\tilde{K}^\bullet, d^\bullet)$ in the derived category of bounded complexes of sheaves of $\mathbb{C}-$vector spaces on X.
Moreover the short exact sequences

$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0 \tag{2}$$

$$0 \to f^*(\hat{\Omega}^1_{\mathbb{C}}) \to (\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to 0. \tag{3}$$

are equivalent to short exact sequences of complexes of left $\tilde{\mathcal{A}}-$modules in the derived category.

Proof. We have to prove that for any $p \geq 0$ the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ are bijective and compatible with the actions of a and b. The case of $\mathcal{H}^p(v)$ is handled (at least for $n \geq 2$ and f reduced) in prop. 2.3.1. of [B.07]. To seek completness and for the convenience of the reader we shall treat here the case of $\mathcal{H}^p(u)$.
First we shall prove the injectivity of $\mathcal{H}^p(u)$. Let $\alpha = df \wedge \beta \in \hat{I}^p \cap Ker\, d$ and assume that we can find $U = \sum_{j=0}^{+\infty} b^j.u_j \in \Omega''^{p-1}[[b]]$ with $\alpha = DU$. Then we have the following relations

$$u_0 = df \wedge \zeta, \quad \alpha = du_0 - df \wedge u_1 \quad \text{and} \quad du_j = df \wedge u_{j+1} \quad \forall j \geq 1.$$

For $j \geq 1$ we have $[du_j] = b[du_{j+1}]$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$; using corollary 2.2. of [B.07] which gives the $b-$separation of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$, this implies $[du_j] = 0, \forall j \geq 1$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$. For instance we can find $\beta_1 \in \hat{K}^{p-1}$ such that $du_1 = d\beta_1$. Now, by de Rham, we can write $u_1 = \beta_1 + d\xi_1$ for $p \geq 2$, where $\xi_1 \in \hat{\Omega}^{p-2}$. Then we conclude that $\alpha = -df \wedge d(\xi_1 + \zeta)$ and $[\alpha] = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$.
For $p = 1$ we have $u_0 = 0$, $du_1 = 0$ so $[\alpha] = [-df \wedge d\xi_1] = 0$ in $\mathcal{H}^1(\hat{I}^\bullet, d^\bullet)$.
We shall show now that the image of $\mathcal{H}^p(u)$ is dense in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its $b-$adic topology. Let $\Omega := \sum_{j=0}^{+\infty} b^j.\omega_j \in \Omega''^p[[b]]$ such that $D\Omega = 0$. The following relations holds $d\omega_j = df \wedge \omega_{j+1} \quad \forall j \geq 0$ and $\omega_0 \in \hat{I}^p$. The corollary 2.2. of [B.07] again allows to find $\beta_j \in \hat{K}^{p-1}$ for any $j \geq 0$ such that $d\omega_j = d\beta_j$. Fix $N \in \mathbb{N}^*$. We have

$$D(\sum_{j=0}^{N} b^j.\omega_j) = b^N.d\omega_N = D(b^N.\beta_N)$$

and $\Omega_N := \sum_{j=0}^N b^j.\omega_j - b^N.\beta_N$ is $D-$closed and in $\Omega''^p[[b]]$. As $\Omega - \Omega_N$ lies in $b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$, the sequence $(\Omega_N)_{N\geq 1}$ converges to Ω in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its $b-$adic topology. Let us show that each Ω_N is in the image of $\mathcal{H}^p(u)$.
Write $\Omega_N := \sum_{j=0}^N b^j.w_j$. The condition $D\Omega_N = 0$ implies $dw_N = 0$ and $dw_{N-1} = df \wedge w_N = 0$. If we write $w_N = dv_N$ we obtain $d(w_{N-1} + df \wedge v_N) = 0$ and $\Omega_N - D(b^N.v_N)$ is of degree $N-1$ in b. For $N = 1$ we are left with $w_0 + b.w_1 - (-df \wedge v_1 + b.dv_1) = w_0 + df \wedge v_1$ which is in $\hat{I}^p \cap Ker\, d$ because $dw_0 = df \wedge dv_1$.
To conclude it is enough to know the following two facts

i) The fact that $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ is complete for its $b-$adic topology.

ii) The fact that $Im(\mathcal{H}^p(u)) \cap b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet) \subset Im(\mathcal{H}^p(u) \circ b^N) \quad \forall N \geq 1$.

Let us first conclude the proof of the surjectivity of $\mathcal{H}^p(u)$ assuming i) and ii).
For any $[\Omega] \in \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ we know that there exists a sequence $(\alpha_N)_{N\geq 1}$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ with $\Omega - \mathcal{H}^p(u)(\alpha_N) \in b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$. Now the property ii) implies that we may choose the sequence $(\alpha_N)_{N\geq 1}$ such that $[\alpha_{N+1}] - [\alpha_N]$ lies in $b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. So the property i) implies that the Cauchy sequence $([\alpha_N])_{N\geq 1}$ converges to $[\alpha] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. Then the continuity of $\mathcal{H}^p(u)$ for the $b-$adic topologies coming from its $b-$linearity, implies $\mathcal{H}^p(u)([\alpha]) = [\Omega]$.
The compatibility with a and b of the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ is an easy exercice.

Let us now prove properties i) and ii).
The property i) is a direct consequence of the completion of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ for its $b-$adic topology given by the corollary 2.2. of [B.07] and the $b-$linear isomorphism $\tilde{b}$ between $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ and $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ constructed in the lemma 2.1.1. above.
To prove ii) let $\alpha \in \hat{I}^p \cap Ker\, d$ and $N \geq 1$ such that

$$\alpha = b^N.\Omega + DU$$

where $\Omega \in \Omega''^p[[b]]$ satisfies $D\Omega = 0$ and where $U \in \Omega''^{p-1}[[b]]$. With obvious notations we have

$$\begin{aligned}
&\alpha = du_0 - df \wedge u_1 \\
&\dots \\
&0 = du_j - df \wedge u_{j+1} \quad \forall j \in [1, N-1] \\
&\dots \\
&0 = \omega_0 + du_N - df \wedge u_{N+1}
\end{aligned}$$

which implies $D(u_0 + b.u_1 + \dots + b^N.u_N) = \alpha + b^N.du_N$ and the fact that du_N lies in $\hat{I}^p \cap Ker\, d$. So we conclude that $[\alpha] + b^N.[du_N]$ is in the kernel of $\mathcal{H}^p(u)$ which is 0. Then $[\alpha] \in b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. ∎

Remark. The map

$$\beta : (\Omega'[[b]]^\bullet, D^\bullet) \to (\Omega''[[b]]^\bullet, D^\bullet)$$

defined by $\beta(\Omega) = b.\Omega$ commutes to the differentials and with the action of b. It induces the isomorphism $\tilde{b}$ of the lemma 2.1.1 on the cohomology sheaves. So it is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]-$modules.
To prove this fact, it is enough to verify that the diagram

$$\begin{tikzcd}
(\hat{K}^\bullet, d^\bullet) \arrow[r, "v"] \arrow[d, "\tilde{b}"] & (\Omega'[[b]]^\bullet, D^\bullet) \arrow[d, "\beta"] \\
(\hat{I}^\bullet, d^\bullet) \arrow[r, "u"] & (\Omega''[[b]]^\bullet, D^\bullet)
\end{tikzcd}$$

induces commutative diagrams on the cohomology sheaves.
But this is clear because if $\alpha = d\xi$ lies in $\hat{K}^p \cap Ker\, d$ we have $D(b.\xi) = b.d\xi - df\wedge\xi$ so $\mathcal{H}^p(\beta)\circ\mathcal{H}^p(v)([\alpha]) = \mathcal{H}^p(u)\circ\mathcal{H}^p(\tilde{b})([\alpha])$ in $\mathcal{H}^p(\Omega''[[b]]^\bullet, D^\bullet)$. ∎

2.3 The existence theorem.

Let us recall some basic definitions on the left modules over the algebra $\tilde{\mathcal{A}}$.

Définition 2.3.1 *An* **(a,b)-module** *is a left $\tilde{\mathcal{A}}-$module which is free and of finite rank on the commutative sub-algebra $\mathbb{C}[[b]]$ of $\tilde{\mathcal{A}}$.*
An (a,b)-module E is

1. **local** *when $\exists N \in \mathbb{N}$ such that $a^N.E \subset b.E$;*
2. **simple pole** *when $a.E \subset b.E$;*
3. **regular** *when it is contained in a simple pole (a,b)-module;*
4. **geometric** *when it is contained in a simple pole (a,b)-module $E^\sharp$ such that the minimal polynomial of the action of $b^{-1}.a$ on $E^\sharp\big/b.E^\sharp$ has its roots in $\mathbb{Q}^{+*}$.*

We shall give more details and examples of (a,b)-modules in the section 3.
Now let E be any left $\tilde{\mathcal{A}}-$module, and define $B(E)$ as the $b-$torsion of E, that is to say

$$B(E) := \{x \in E \ / \ \exists N \quad b^N.x = 0\}.$$

Define $A(E)$ as the $a-$torsion of E and

$$\hat{A}(E) := \{x \in E \ / \ \mathbb{C}[[b]].x \subset A(E)\}.$$

Remark that $B(E)$ and $\hat{A}(E)$ are sub-$\tilde{\mathcal{A}}-$modules of E but that $A(E)$ is not stable by b.

Définition 2.3.2 *A left $\tilde{\mathcal{A}}-$module E is* **small** *when the following conditions hold*

1. E *is a finite type* $\mathbb{C}[[b]]-$*module ;*

2. $B(E) \subset \hat{A}(E)$ *;*

3. $\exists N \ / \ a^N.\hat{A}(E) = 0$ *;*

Recall that for E small we have always the equality $B(E) = \hat{A}(E)$ and that this complex vector space is finitie dimensional. The quotient $E/B(E)$ is an (a,b)-module called **the associate (a,b)-module** to E.
Conversely, any left $\tilde{\mathcal{A}}-$module E such that $B(E)$ is a finite dimensional $\mathbb{C}-$vector space and such that $E/B(E)$ is an (a,b)-module is small.
The following easy criterium to be small will be used later :

Lemme 2.3.3 *A left* $\tilde{\mathcal{A}}-$*module* E *is small if and only if the following conditions hold :*

1. $\exists N \ / \ a^N.\hat{A}(E) = 0$ *;*

2. $B(E) \subset \hat{A}(E)$ *;*

3. $\cap_{m \geq 0} b^m.E \subset \hat{A}(E)$ *;*

4. $Ker\, b$ *and* $Coker\, b$ *are finite dimensional complex vector spaces.*

As the condition 3) in the previous lemma has been omitted in [B.07] (but this does not affect the results of this article because this lemma was used only in a case where this condition 3) was satisfied, thanks to proposition 2.2.1. of *loc. cit.*), we shall give the (easy) proof.

PROOF. First the conditions 1) to 4) are obviously necessary. Conversely, assume that E satisfies these four conditions. Then condition 2) implies that the action of b on $\hat{A}(E)/B(E)$ is injective. But the condition 1) implies that $b^{2N} = 0$ on $\hat{A}(E)$ (see [B.06]). So we conclude that $\hat{A}(E) = B(E) \subset Ker\, b^{2N}$ which is a finite dimensional complex vector space using condition 4) and an easy induction. Now $E/B(E)$ is a $\mathbb{C}[[b]]-$module which is separated for its $b-$adic topology. The finitness of $Coker\, b$ now shows that it is a free finite type $\mathbb{C}[[b]]-$module concluding the proof. ■

Définition 2.3.4 *We shall say that a left* $\tilde{\mathcal{A}}-$*module* E *is* **geometric** *when* E *is small and when it associated (a,b)-module* $E/B(E)$ *is geometric.*

The main result of this section is the following theorem, which shows that the Gauss-Manin connexion of a proper holomorphic function produces geometric $\tilde{\mathcal{A}}-$modules associated to vanishing cycles and nearby cycles.

Théorème 2.3.5 *Let X be a connected complex manifold of dimension $n+1$ where $n \in \mathbb{N}$, and let $f : X \to D$ be an non constant proper holomorphic function to an open disc D in $\mathbb{C}$ with center 0. Let us assume that df is nowhere vanishing outside of $X_0 := f^{-1}(0)$.*
Then the $\tilde{\mathcal{A}}-$modules

$$\mathbb{H}^j(X, (\hat{K}^\bullet, d^\bullet)) \quad \text{and} \quad \mathbb{H}^j(X, (\hat{I}^\bullet, d^\bullet))$$

are geometric for any $j \geq 0$.

In the proof we shall use the $\mathcal{C}^\infty$ version of the complex $(\hat{K}^\bullet, d^\bullet)$. We define K^p_∞ as the kernel of $\wedge df : \mathcal{C}^{\infty,p} \to \mathcal{C}^{\infty,p+1}$ where $\mathcal{C}^{\infty,j}$ denote the sheaf of $\mathcal{C}^\infty-$forms on X of degree j, let $\hat{K}^p_\infty$ be its formal $f-$completion and $(\hat{K}^\bullet_\infty, d^\bullet)$ the corresponding de Rham complex.
The next lemma is proved in [B.07] (lemma 6.1.1.)

Lemme 2.3.6 *The natural inclusion*

$$(\hat{K}^\bullet, d^\bullet) \hookrightarrow (\hat{K}^\bullet_\infty, d^\bullet)$$

is a quasi-isomorphism.

Remark. As the sheaves $\hat{K}^\bullet_\infty$ are fine, so we have a natural isomorphism

$$\mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \simeq H^p\big(\Gamma(X, \hat{K}^\bullet_\infty), d^\bullet\big).$$

Let us denote by X_1 the generic fiber of f. Then X_1 is a smooth compact complex manifold of dimension n and the restriction of f to $f^{-1}(D^*)$ is a locally trivial $\mathcal{C}^\infty$ bundle with typical fiber X_1 on $D^* = D \setminus \{0\}$, if the disc D is small enough around 0. Fix now $\gamma \in H_p(X_1, \mathbb{C})$ and let $(\gamma_s)_{s \in D^*}$ the corresponding multivalued horizontal family of $p-$cycles $\gamma_s \in H_p(X_s, \mathbb{C})$. Then for $\omega \in \Gamma(X, \hat{K}^p_\infty \cap Ker\, d)$ define the multivalued holomorphic function

$$F_\omega(s) := \int_{\gamma_s} \frac{\omega}{df}.$$

Let now

$$\Xi := \sum_{\alpha \in \mathbb{Q} \cap]-1,0],\, j \in [0,n]} \mathbb{C}[[s]].s^\alpha.\frac{(Log s)^j}{j!}.$$

This is an $\tilde{\mathcal{A}}-$modules with a acting as multiplication by s and b as the primitive in s without constant. Now if $\hat{F}_\omega$ is the asymptotic expansion at 0 of F_ω, it is an element in Ξ, and we obtain in this way an $\tilde{\mathcal{A}}-$linear map

$$Int : \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \to H^p(X_1, \mathbb{C}) \otimes_{\mathbb{C}} \Xi.$$

To simplify notations, let $E := \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet))$. Now using Grothendieck theorem [G.66], there exists $N \in \mathbb{N}$ such that $Int(\omega) \equiv 0$, implies $a^N.[\omega] = 0$ in E.

As the converse is clear we conclude that $\hat{A}(E) = Ker(Int)$. It is also clear that $B(E) \subset Ker(Int)$ because Ξ has no $b-$torsion. So we conclude that E satisfies properties 1) and 2) of the lemma 2.3.3.
The property 3) is also true because of the regularity of the Gauss-Manin connexion of f.

End of the proof of theorem 2.3.5. To show that $E := \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet))$ is small, it is enough to prove that E satisfies the condition 4) of the lemma 2.3.3. Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0.$$

It contains the exact sequence

$$\mathbb{H}^{p-1}(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \to \mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet)) \overset{\mathbb{H}^p(i)}{\to} \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \to \mathbb{H}^p(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$$

and we know that b is induced on the complex of $\tilde{\mathcal{A}}-$modules quasi-isomorphic to $(\hat{K}^\bullet, d^\bullet)$ by the composition $i \circ \tilde{b}$ where $\tilde{b}$ is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]-$modules. This implies that the kernel and the cokernel of $\mathbb{H}^p(i)$ are isomorphic (as $\mathbb{C}-$vector spaces) to $Ker\, b$ and $Coker\, b$ respectively. Now to prove that E satisfies condition 4) of the lemma 2.3.3 it is enough to prove finite dimensionality for the vector spaces $\mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$ for all $j \geq 0$.
But the sheaves $[\hat{K}/\hat{I}]^j \simeq [Ker\, df/Im\, df]^j$ are coherent on X and supported in X_0. The spectral sequence

$$E_2^{p,q} := H^q\big(H^p(X, [\hat{K}/\hat{I}]^\bullet), d^\bullet\big)$$

which converges to $\mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$, is a bounded complex of finite dimensional vector spaces by Cartan-Serre. This gives the desired finite dimensionality.
To conclude the proof, we want to show that $E/B(E)$ is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of f and of the Monodromy theorem, which are already incoded in the definition of Ξ : the injectivity on $E/B(E)$ of the $\tilde{\mathcal{A}}-linear$ map Int implies that $E/B(E)$ is geometric. Remark now that the piece of exact sequence above gives also the fact that $\mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet))$ is geometric, because it is an exact sequence of $\tilde{\mathcal{A}}-$modules. ■

3 Basic properties.

3.1 Definition and examples.

First recall in a more naïve way the definition of an (a,b)-module.

Définition 3.1.1 *An* **(a,b)-module** E *is a free finite type* $\mathbb{C}[[b]]-$*module with a* $\mathbb{C}-$*linear endomorphism* $a : E \to E$ *which is continuous for the* $b-$*adic topology of* E *and satisfies* $a.b - b.a = b^2$.
The **rank of** E, *denote by* $rank(E)$, *will be the rank of* E *as a* $\mathbb{C}[[b]]-$*module.*

REMARKS.

1. Let $(e_1, \cdots, e_k)$ a $\mathbb{C}[[b]]-$basis of a free finite type $\mathbb{C}[[b]]-$module. Then choosing arbitrarily elements $(\varepsilon_1, \cdots, \varepsilon_k)$ and defining $a.e_j = \varepsilon_j \quad \forall j \in [1, k]$ gives an (a,b)-module: the commutation relation implies that $\forall n \in \mathbb{N}$ we have $a.b^n = b^n.a + n.b^{n+1}$ so a is defined on $\sum_{j=1}^k \quad \mathbb{C}[b].e_j$. The continuity assumption gives its (unique) extension.

2. There is a natural (a,b)-module associated to every algebraic linear differential system (see [B.95] p.42)

$$Q(z).\frac{dF}{dz} = M(z).F(z), \quad Q \in \mathbb{C}[z], \quad M \in End(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathbb{C}[z].$$

In the sequel of this article we shall mainly consider regular (a,b)-modules (see definition recalled below). To try to convince the reader that the "general" (a,b)-module structure is interesting, let me quote the following result, which is quite elementary in the regular case, but which is not so easy in general.

Théorème 3.1.2 *([B.95] th.1bis p.31) Let E be an (a,b)-module. Then the kernel and cokernel of "a" are finite dimensional.*

This result implies a general finiteness theorem for extensions of (a,b)-modules (see [B.95] and also section 1.3).

Définition 3.1.3 *We shall say that an (a,b)-module E has a* **simple pole** *when the inclusion $a.E \subset b.E$ is satisfied.*

This terminology comes from the terminology of meromorphic connexions (see for instance [D.70]).

EXAMPLE. For any $\lambda \in \mathbb{C}$ define the simple pole rank 1 (a,b)-module E_λ as $E := \mathbb{C}[[b]].e_\lambda$ where "a" is defined by the relation $a.e_\lambda = \lambda.b.e_\lambda$. $\square$

As an introduction to our second theorem, the reader may solve the following exercice by direct computation.

EXERCICE. For any $S \in \mathbb{C}[[b]]$ show that the simple pole (a,b)-module defined by $E := \mathbb{C}[[b]].e_S$ and $a.e_S = b.S(b).e_S$ is isomorphic to E_λ with $\lambda = S(0)$ (hint: begin by looking for $\alpha_1 \in \mathbb{C}$ such that $(a - S(0).b)(e + \alpha_1.b.e) \in b^3.E)$. $\square$

For a simple pole (a,b)-module, the linear map $b^{-1}.a : E \to E$ is well defined and induces an endomorphism $f := b^{-1}.a : E/b.E \to E/b.E$. For any $\lambda \in \mathbb{C}$ we shall denote by λ_{min} the smallest eigenvalue of f which is in $\lambda + \mathbb{Z}$. Then for

$\lambda = \lambda_{min} - k$ with $k \in \mathbb{N}^*$ the bijectivity of the map $f - \lambda$ on $E/b.E$ implies easily its bijectivity on E (see the exercice above). It gives then the equality

$$(a - \lambda.b).E = b.E.$$

Using this remark, it is not difficult to prove the following result from [B.93] (prop.1.3. p.11) that we shall use later on.

Proposition 3.1.4 *Let E be a simple pole (a,b)-module, and let $\lambda \in \mathbb{C}$ and $\kappa \in \mathbb{N}$ such that $\lambda - \kappa \leq \lambda_{min}$. If $y \in E$ satisfies $(a - \lambda.b).y \in b^{\kappa+2}.E$ then there exists an unique $\tilde{y} \in E$ such that $(a - \lambda.b).\tilde{y} = 0$ and $\tilde{y} - y \in b^{\kappa+1}.E$.*

An easy consequence of this proposition is that for an eigenvalue λ of f such that $\lambda = \lambda_{min}$ there always exists a non zero $x \in E$ such that $(a - \lambda.b).x = 0$. This gives an embedding of E_λ in E. Remark also that if E is a non zero simple pole (a,b)-module, such a λ always exists. This leads to a rather precise description a of "general" simple pole (a,b)-module (see [B.93] th. 1.1 p.15).

Définition 3.1.5 *An (a.b)-module E is* **regular** *when its saturation by $b^{-1}.a$ in $E[b^{-1}]$ is finitely generated on $\mathbb{C}[[b]]$.*

We shall denote $E^\sharp$ this saturation. It is a simple pole (a,b)-module and it is the smallest simple pole (a,b)-module containing E in the sense that for any (a,b)-linear morphism $j : E \to F$ where F is a simple pole (a,b)-module, there exists a unique (a,b)-linear extension $j^\sharp : E^\sharp \to F$ of j.

It is easy to show that a regular (a,b)-module of rank 1 is isomorphic to some E_λ for some $\lambda \in \mathbb{C}$. The classification of rank 2 regular (a,b)-module is not so obvious. We recall it here for a later use

Proposition 3.1.6 *(see [B.93] prop.2.4 p. 34) The list of rank 2 regular (a,b)-modules is, up to isomorphism, the following :*

1. *$E_\lambda \oplus E_\mu$ for $(\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2$.*
2. *For any $\lambda \in \mathbb{C}$ and any $n \in \mathbb{N}$ let $E_\lambda(n)$ be the simple pole (a,b)-module with basis (x, y) such that*

$$a.x = (\lambda + n).b.x + b^{n+1}.y \quad \text{and} \quad a.y = \lambda.b.y.$$

3. *For any $(\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2$ let $E_{\lambda,\mu}$ the rank 2 regular (a,b)-module with basis (y, t) such that*

$$a.y = \mu.b.y \quad \text{and} \quad a.t = y + (\lambda - 1).b.t.$$

4. *For any* $\lambda \in \mathbb{C}$, *any* $n \in \mathbb{N}^*$ *and any* $\alpha \in \mathbb{C}^*$ *let* $E_{\lambda,\lambda-n}(\alpha)$ *be the rank 2 regular (a,b)-module with basis* (y,t) *such that*

$$a.y = (\lambda - n).b.y \quad \text{and} \quad a.t = y + (\lambda - 1)b.t + \alpha.b^n.y$$

Note that the first two cases are simple pole (a,b)-modules.
The saturation by $b^{-1}.a$ in case 3. is generated by $b^{-1}.y$ and t as a $\mathbb{C}[[b]]-$module. It is isomorphic to $E_{\lambda-1} \oplus E_{\mu-1}$ for $\lambda \neq \mu$ and to $E_{\lambda-1}(0)$ for $\lambda = \mu$.
The saturation by $b^{-1}.a$ in case 4. is generated by $b^{-1}.y$ and t as a $\mathbb{C}[[b]]-$module. It is isomorphic to $E_{\lambda-n-1}(n)$ for any non zero value of α.

To conclude this first section, let me recall also the theorem of existence of Jordan-Hölder sequences for regular (a,b)-module, which will be usefull in the induction in the proof of our result .

Théorème 3.1.7 *(see [B.93] th. 2.1 p.30) For any regular rank k (a,b)-module* E *there exists a sequence of sub-(a,b)-modules*

$$0 = E^0 \subset E^1 \subset \cdots \subset E^{k-1} \subset E^k = E$$

such that for any $j \in [1,k]$ *the quotient* E^j/E^{j-1} *is isomorphic to* E_{λ_j}. *Moreover we may choose for* E^1 *any normal*[1] *rank 1 sub-(a,b)-module of* E.
The number $\alpha(E) := \sum_{j=1}^k \lambda_j$ *is independant of the choice of the Jordan-Hölder sequence. It is given by the following formula*

$$\alpha(E) = trace\big(b^{-1}.a : E^\sharp/b.E^\sharp \to E^\sharp/b.E^\sharp\big) + \dim_{\mathbb{C}}(E^\sharp/E).$$

3.2 The regularity order.

Définition 3.2.1 *Let* E *be a regular (a,b)-module. We define the* **regularity order of** E *as the smallest integer* $k \in \mathbb{N}$ *such that the inclusion*

$$a^{k+1}.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}.E \tag{reg.}$$

is valid. We shall note this integer $or(E)$.
We define also **the index** $\delta(E)$ **of** E *as the smallest integer* $m \in \mathbb{N}$ *such that* $E^\sharp \subset b^{-m}.E$.

Remarks.

i) The (a,b)-module E has a simple pole if an only iff $or(E) = 0$.

[1]normal means $E^1 \cap b.E = b.E^1$, so that E/E^1 is again free on $\mathbb{C}[[b]]$.

ii) The inclusion (reg.) implies that $(b^{-1}.a)^{k+1}.E \subset \Phi_k(E) := \sum_{j=0}^{k} \quad (b^{-1}.a)^j.E$ and this implies that $\Phi_k(E)$ is stable by $b^{-1}.a$. So $\Phi_k(E)$ is a simple pole (a,b)-module contained in $b^{-k}.E \subset E[b^{-1}]$. This implies clearly the regularity of E.
For $k = or(E)$ we have $E^\sharp = \Phi_k(E) \subset b^{-k}.E$. So we have $\delta(E) \leq or(E)$.

iii) As the quotient $b^{-k}.E/E$ is a finite dimensional $\mathbb{C}-$vector space, the quotient $E^\sharp/E$ is always a finite dimensional $\mathbb{C}-$vector space. $\square$

The remark iii) shows that for a regular (a,b)-module E there always exists a simple pole sub-(a,b)-module of E which is a finite codimensional vector space in E. This comes from the fact that for $k = \delta(E)$ we have $b^k.E^\sharp \subset E$ and that $b^k.E^\sharp$ has a simple pole.

Example. The inequality $\delta(E) \leq or(E)$ may be strict for $or(E) \geq 2$. For instance the (a,b)-module of rank 3 with $\mathbb{C}[[b]]-$basis e_1, e_2, e_3 with $a.e_1 = e_2, \quad a.e_2 = b.e_3, \quad a.e_3 = 0$ has index 1 and regularity order 2 : an easy computation gives that a $\mathbb{C}[[b]]-$basis for $E^\sharp$ is given by $e_1, b^{-1}.e_2, b^{-1}.e_3$, and that a $\mathbb{C}[[b]]-$basis for $E + b^{-1}.a.E$ is given by $e_1, b^{-1}.e_2, e_3$. $\square$

Définition 3.2.2 *Let E be a regular (a,b)-module. The* **biggest simple pole sub-(a,b)-module of** E *exists and has finite $\mathbb{C}-$codimension in E. We shall note it E^b.*

In general, for $k = \delta(E)$ the inclusion $b^k.E^\sharp \subset E^b$ is strict. For instance this is the case for $E_{\lambda,\mu} \oplus E_\nu$.

Lemme 3.2.3 *Let E be a regular (a,b)-module. The smallest integer m such we have $b^m.E \subset E^b$ is equal to $\delta(E)$.*

Proof. Let $k := \delta(E)$. Then $b^k.E^\sharp$ is a simple pole sub-(a,b)-module of E. So we have $b^k.E \subset b^k.E^\sharp \subset E^b$. Conversely, an inclusion $b^m.E \subset E^b$ gives $E \subset b^{-m}.E^b$. As $b^{-m}.E^b$ has a simple pole this implies $E^\sharp \subset b^{-m}.E^b \subset b^{-m}.E$. So $\delta(E) \leq m$. $\blacksquare$

Examples. In the case 3 of the proposition 3.1.6 E^b is generated as a $\mathbb{C}[[b]]-$module by y and $b.t$, so $E^b = b.E^\sharp$.
In case 4 we have also $E^b = b.E^\sharp$.

Lemme 3.2.4 *Let E be a regular (a,b)-module. For any exact sequence of (a,b)-modules*

$$0 \to E' \to E \xrightarrow{\pi} E'' \to 0 \qquad (*)$$

we have $or(E'') \leq or(E) \leq rank(E') + or(E'')$.
As a consequence, the order of regularity of E is at most $rank(E) - 1$ for any regular non zero (a,b)-module.

PROOF. The inequality $or(E'') \leq or(E)$ is trivial because an inequality

$$a^{k+1}.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}.E$$

implies the same for E'' and, by definition, the best such integer k is the order of regularity.

The crucial case is when E' is of rank 1 . So we may assume that $E' \simeq E_\lambda$ for some $\lambda \in \mathbb{C}$ (see 3.1.7 or [B.93] prop.2.2 p.23). Let $k = or(E'')$. Then the inclusion

$$a^{k+1}.E'' \subset \sum_{j=0}^{k} a^j.b^{k-j+1}.E'' \tag{1}$$

implies that

$$a^{k+1}.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}.E + b^l.E_\lambda \tag{2}$$

for some $l \in \mathbb{N}$. In fact we can take for l the smallest integer such that the generator e_λ of E_λ (defined up to $\mathbb{C}^*$ by the relation $a.e_\lambda = \lambda.b.e_\lambda$) satisfies $b^l.e_\lambda \in \Psi_k = \sum_{j=0}^{k} a^j.b^{k-j+1}.E$.

Remark that this integer $l \geq 0$ is well defined because $b^{k+1}.e_\lambda \in \Psi_k$. Moreover, as Ψ_k is a $\mathbb{C}[[b]]-$submodule of E, $b^l.e_\lambda \in \Psi_k$ implies $b^l.E_\lambda \subset \Psi_k$.

Now, thanks to (2) we have

$$a^{k+2}.E \subset \sum_{j=0}^{k} a^{j+1}.b^{k+1-j}.E \quad + a.b^l.E_\lambda \tag{3}$$

which gives

$$a^{k+2}.E \subset \sum_{j=0}^{k+1} a^j.b^{k-j+2}.E \tag{4}$$

because $a.b^l.E_\lambda = b.b^l.E_\lambda \subset b.\Psi_k$.

This proves that $or(E)$ is at most $k+1 = or(E'') + rank(E')$.

Assume now that our inequality is proved for E' of rank $p-1$ and consider an exact sequence $(^*)$ with $rank(E')$ equal $p \geq 2$. Let $E_\lambda \subset E'$ be a normal rank 1 sub-(a,b)-module of E' (see 3.1.7 or [B.93] prop.2.2 p.23 for a proof of the existence of such sub-(a,b)-module) and consider the exact sequence of (a,b)-modules (using the fact that E_λ is also normal in E; see lemma 2.5 of [B.93])

$$0 \to E'/E_\lambda \to E/E_\lambda \to E'' \to 0$$

Using the induction hypothesis and the rank 1 case we get

$$or(E) \leq or(E/E_\lambda) + 1 \leq p - 1 + or(E'') + 1 = p + or(E'').$$

Now using an easy induction (or a Jordan-Hölder sequence for E) we obtain $or(E) \leq rank(E) - 1$ for any regular E. ■

Remark. In the situation of the previous lemma we have $\delta(E') \leq \delta(E)$. This is a consequence of the obvious inclusion $(E')^\sharp \subset E'[b^{-1}] \cap E^\sharp$: assume that $x \in E'[b^{-1}] \cap E^\sharp$; then, for $k := \delta(E)$ we have $b^k.x \in E'[b^{-1}] \cap E$ so that $b^{N+k}.x \in E'$ for N large enough. As E/E' has no $b-$torsion, we conclude that $b^k.x \in E'$. So our initial inclusion implies $\delta(E') \leq k$. $\square$

3.3 Duality.

In this section we consider more carefully the associative and unitary $\mathbb{C}-$algebra

$$\tilde{\mathcal{A}} := \big\{ \sum_0^\infty P_n(a).b^n \quad \text{with} \quad P_n \in \mathbb{C}[z] \big\}$$

with the commutation relation $a.b - b.a = b^2$, and such that the left and right multiplications by a are continuous for the $b-$adic topology[2] of $\tilde{\mathcal{A}}$.

The right structure as a commuting left-structure on $\tilde{\mathcal{A}}$.
There exits an unique $\mathbb{C}-$linear (bijective) map $\theta : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ with the following properties

i) $\theta(1) = 1, \quad \theta(a) = a, \quad \theta(b) = -b;$

ii) $\theta(x.y) = \theta(y).\theta(x) \quad \forall x, y \in \tilde{\mathcal{A}}.$

iii) θ is continuous for the $b-$adic topology of $\tilde{\mathcal{A}}$

The uniqueness is an easy consequence of iii) and the fact that the conditions i) and ii) implies $\theta(b^p.a^q) = (-1)^p.a^q.b^p \quad \forall p, q \in \mathbb{N}$. Existence is then clear from the explicit formula deduced from this remark.
We define a new structure of left $\tilde{\mathcal{A}}-$module on $\tilde{\mathcal{A}}$, **called the $\theta-$structure** and denote by $x_*\square$, by the formula

$$x_*y = y.\theta(x).$$

It is easy to see that this new left-structure on $\tilde{\mathcal{A}}$ commutes with the ordinary one and that with this $\theta-$structure $\tilde{\mathcal{A}}$ is still free of rank one as a left $\tilde{\mathcal{A}}-$module.

Définition 3.3.1 *Let E be a (left) $\tilde{\mathcal{A}}-$module. On the $\mathbb{C}-$vector space $Hom_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ we define a left $\tilde{\mathcal{A}}-$module structure using the $\theta-$structure on $\tilde{\mathcal{A}}$. Explicitely this means that for $\varphi \in Hom_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ and $x \in \tilde{\mathcal{A}}$ we let*

$$\forall e \in E \quad (x.\varphi)(e) := x_*\varphi(e) = \varphi(e).\theta(x).$$

We obtain in this way a left $\tilde{\mathcal{A}}-$module that we shall still denote $Hom_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$.

[2] remark that for each $k \in \mathbb{N}$ $b^k.\tilde{\mathcal{A}} = \tilde{\mathcal{A}}.b^k$.

It is clear that $E \to Hom_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ is a contravariant functor which is left exact in the category of left $\tilde{\mathcal{A}}-$modules. As every finite type left $\tilde{\mathcal{A}}-$module has a resolution of length ≤ 2 by free finite type modules (see [B.95] cor.2 p.29), we shall denote by $Ext^i_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}}), i \in [0,2]$ the right derived functors of this functor. They are finite type left $\tilde{\mathcal{A}}-$modules when E is finitely generated because $\tilde{\mathcal{A}}$ is left noetherian (see [B.95] prop.2 p.26).

Any (a,b)-module is a left $\tilde{\mathcal{A}}-$module. They are characterized by the existence of special simple resolutions.

Lemme 3.3.2 *Let M be a (p,p) matrix with entries in the ring $\mathbb{C}[[b]]$. Then the left $\tilde{\mathcal{A}}-$linear map $Id_p.a - M : \tilde{\mathcal{A}}^p \to \tilde{\mathcal{A}}^p$ given by*

$$^tX := (x_1, \cdots, x_p) \ \to \ ^tX.(Id_p.a - M)$$

is injective. Its cokernel is the (a,b)-module E given as follows :
E has a $\mathbb{C}[[b]]$ base $e := (e_1, \cdots, e_p)$ and a is defined by the two conditions

1. *$a.e := M(b).e$;*

2. *the left action of a is continuous for the $b-$adic topology of E.*

Any (a,b)-module is obtained in this way and so, as a $\tilde{\mathcal{A}}-$left-module, has a resolution of the form

$$0 \to \tilde{\mathcal{A}}^p \overset{^t\square.(Id_p.a-M)}{\longrightarrow} \tilde{\mathcal{A}}^p \to E \to 0. \qquad (@)$$

Proof. First remark that for $x \in \tilde{\mathcal{A}}$ the condition $x.a \in b.\tilde{\mathcal{A}}$ implies $x \in b.\tilde{\mathcal{A}}$. Now let us prove, by induction on $n \geq 1$, that, for any (p,p) matrix M with entries in $\mathbb{C}[[b]]$ the condition $^tX.(Id_p.a - M) = 0$ implies $^tX \in b^n.\tilde{\mathcal{A}}^p$.
For $n = 1$ this comes from the previous remark. Let assume that the assertion is proved for $n \geq 1$ and consider an $X \in \tilde{\mathcal{A}}^p$ such that $^tX.(Id_p.a - M) = 0$. Using the induction hypothesis we can find $Y \in \tilde{\mathcal{A}}^p$ such that $X = b^n.Y$. Now we obtain, using $a.b^n = b^n.a + n.b^{n+1}$ and the fact that $\tilde{\mathcal{A}}$ has no zero divisor, the relation

$$^tY(Id_p.a - (M + n.Id_p.b)) = 0$$

and using again our initial remark we conclude that $Y \in b.\tilde{\mathcal{A}}^p$ so $X \in b^{n+1}.\tilde{\mathcal{A}}^p$.
So such an X is in $\cap_{n\geq 1} b^n.\tilde{\mathcal{A}}^p = (0)$.
The other assertions of the lemma are obvious. ■

We recall now a construction given in [B.95] which allows to compute more easily the vector spaces $Ext^i_{\tilde{\mathcal{A}}}(E, F)$ when E, F are (a,b)-modules

Définition 3.3.3 *Let E, F two (a,b)-modules. Then the $\mathbb{C}[[b]]-$module $Hom_b(E, F)$ is again a free and finitely generated $\mathbb{C}[[b]]-$module. Define on it an (a,b)-module structure in the following way.*

1. *First change the sign of the action of* b. *So* $S(b) \in \mathbb{C}[[b]]$ *will act as* $\check{S}(b) = S(-b)$.

2. *Define* a *using the linear map* $\Lambda : Hom_b(E,F) \to Hom_b(E,F)$ *given by* $\Lambda(\varphi)(e) = \varphi(a.e) - a.\varphi(e)$.

We shall denote $Hom_{a,b}(E,F)$ *the corresponding (a,b)-module.*

The verification that $\Lambda(\varphi)$ is $\mathbb{C}[[b]]-$linear and that $\Lambda.\check{b} - \check{b}.\Lambda = \check{b}^2$ are easy (and may be found in [B.95] p.31).

Remark. In *loc. cit.* we defined the (a,b)-module structure on $Hom_{a,b}(E,F)$ with opposite signs for a and b. The present convention is better because it fits with the usual definition of the formal adjoint of a differential operator : $z^* = z$ and $(\partial/\partial z)^* = -\partial/\partial z$. $\square$

The following lemma is also proved in *loc.cit.*

Lemme 3.3.4 *Let* E, F *two (a,b)-modules. Then there is a functorial isomorphism of* $\mathbb{C}-$*vector spaces*

$$H^i\Big(Hom_{a,b}(E,F) \xrightarrow{a} Hom_{a,b}(E,F)\Big) \to Ext^i_{\tilde{\mathcal{A}}}(E,F) \quad \forall i \geq 0.$$

Here the map a *of the complex* $Hom_{a,b}(E,F) \xrightarrow{a} Hom_{a,b}(E,F))$ *is equal to the* Λ *defined above which is, by definition, the operator "a" of the (a,b)-module* $Hom_{a,b}(E,F)$.

Now the following corollary of the lemma 3.3.2 gives that the two natural ways of defining the dual of an (a,b)-module give the same answer.

Corollaire 3.3.5 *Let* E *an (a,b)-module. There is a functorial isomorphism of (a,b)-modules between the following two (a,b)-modules constructed as follows :*

1. $Ext^1_{\tilde{\mathcal{A}}}(E,\tilde{\mathcal{A}})$ *with the* $\tilde{\mathcal{A}}-$*structure defined by the* $\theta-$*structure of* $\tilde{\mathcal{A}}$.

2. $Hom_{a,b}(E,E_0)$ *where* $E_0 := \tilde{\mathcal{A}}\big/\tilde{\mathcal{A}}.a$.

Proof. Using a free resolution (@) of E deduced from a $\mathbb{C}[[b]]-$basis $e := (e_1, \cdots, e_p)$ we obtain, by the previous lemma, an exact sequence

$$0 \to \tilde{\mathcal{A}}^p \overset{(Id_p.a - {}^tM).\square}{\longrightarrow} \tilde{\mathcal{A}}^p \to Ext^1_{\tilde{\mathcal{A}}}(E,\tilde{\mathcal{A}}) \to 0. \qquad (@@)$$

of left $\tilde{\mathcal{A}}-$modules where $\tilde{\mathcal{A}}^p$ is endowed with its $\theta-$structure. Writing the same exact sequence with the ordinary left-module structure of $\tilde{\mathcal{A}}^p$ gives

$$0 \to \tilde{\mathcal{A}}^p \overset{{}^t\square.(Id_p.a - {}^t\check{M})}{\longrightarrow} \tilde{\mathcal{A}}^p \to Ext^1_{\tilde{\mathcal{A}}}(E,\tilde{\mathcal{A}}) \to 0. \qquad (@@ \text{ bis})$$

where ${}^t\check{M}(b) := {}^tM(-b)$.
Denote by $e^* := (e_1^*, \cdots, e_p^*)$ the dual basis of $Hom_{\mathbb{C}[[b]]}(E, E_0)$. By definition of the action of a on $Hom_{a,b}(E, E_0)$ we get, if ω is the class of 1 in E_0 :

$$(a.e_i^*)(e_j) = e_i^*(a.e_j) - a.e_i^*(e_j) = e_i^*(\sum_{h=1}^{p} m_{j,h}.e_h) - a.\delta_{i.j}.\omega = \check{m}_{j,i}.\omega$$

because $a.\omega = 0$ in E_0, and the definition of the action of b on $Hom_{a,b}(E, E_0)$.
So we have $a.e^* =^t \check{M}.e^*$ concluding the proof. ■

Définition 3.3.6 *For any (a,b)-module E the* **dual of** *E, denoted by E^*, is the (a,b)-module $Ext^1_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}}) \simeq Hom_{a,b}(E, E_0)$.*

Of course, for any $\tilde{\mathcal{A}}-$linear map $f : E \to F$ between two (a,b)-modules we have an $\tilde{\mathcal{A}}-$linear "dual" map $f^* : F^* \to E^*$.
It is an easy consequence of our previous description of $Ext^1_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ that we have a functorial isomorphism $(E^*)^* \to E$.

Examples.

1. For each $\lambda \in \mathbb{C}$ we have $(E_\lambda)^* \simeq E_{-\lambda}$.
2. For $(\lambda, \mu) \in \mathbb{C}^2$ we have $E^*_{\lambda,\mu} \simeq E_{-\mu+1,-\lambda+1}$.
3. Let E be the rank two simple pole (a,b)-module $E_1(0)$ defined by $a.e_1 = b.e_1 + b.e_2$ and $a.e_2 = b.e_2$. Then its dual is isomorphic to $E_{-1}(0)$.
   It is also an elementary exercice to show the following isomorphisms :

   $$E_1(0) \simeq \mathbb{C}[[z]] \oplus \mathbb{C}[[z]].Logz \quad \text{and} \quad E_{-1}(0) \simeq \mathbb{C}[[z]]\frac{1}{z^2} \oplus \mathbb{C}[[z]].\frac{Logz}{z^2}$$

   with $a := \times z$ and $b := \int_0^z$.

Proposition 3.3.7 *For any exact sequence of (a,b)-modules*

$$0 \to E' \xrightarrow{u} E \xrightarrow{v} E'' \to 0$$

we have an exact sequence of (a,b)-modules

$$0 \to (E'')^* \xrightarrow{v^*} E^* \xrightarrow{u^*} (E')^* \to 0.$$

If E is a simple pôle (a,b)-module, E^* has a simple pole.
For any regular (a,b)-module E its dual E^* is regular. Moreover, if E^b and $E^\sharp$ are respectively the biggest simple pole submodule of E and the saturation of E by $b^{-1}.a$ in $E[b^{-1}]$, we have

$$(E^\sharp)^* \simeq (E^*)^b \quad \text{and} \quad (E^b)^* \simeq (E^*)^\sharp.$$

PROOF. The first assertion is a direct consequence of the vanishing of $Ext^i_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ for $i = 0, 2$, for any (a,b)-module and the long exact sequence for the "Ext".
The condition that E has a simple pole is equivalent to the fact that for any choosen basis e of E the matrix M has its coefficients in $b.\tilde{\mathcal{A}} = \tilde{\mathcal{A}}.b$. Then this remains true for ${}^t\check{M}$.
To prove the regularity of E^* when E is regular, we shall use induction on the rank of E. The rank 1 case is obvious because we have a simple pole in this case. Assume that the assertion is true for $rank < p$ and consider a $rank = p$ regular (a,b)-module E. Using the theorem 3.1.7 we have an exact sequence of (a,b)-modules

$$0 \to E_\lambda \to E \to F \to 0$$

where F is regular of rank $p - 1$. This gives a short exact sequence

$$0 \to F^* \to E^* \to E_{-\lambda} \to 0$$

and the regularity of F^* and of $E_{-\lambda}$ implies the regularity of E^*.
Now the inclusions $E^b \subset E \subset E^\sharp$ gives exact sequences

$$0 \to Ext^1_{\tilde{\mathcal{A}}}(E/E^b, \tilde{\mathcal{A}}) \to E^* \to (E^b)^* \to Ext^2_{\tilde{\mathcal{A}}}(E/E^b, \tilde{\mathcal{A}}) \to 0$$
$$0 \to Ext^1_{\tilde{\mathcal{A}}}(E^\sharp/E, \tilde{\mathcal{A}}) \to (E^\sharp)^* \to E^* \to Ext^2_{\tilde{\mathcal{A}}}(E^\sharp/E, \tilde{\mathcal{A}}) \to 0$$

and the next lemma will show that the $Ext^1_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}}) = 0$ for any $\tilde{\mathcal{A}}-$module which is a finite dimensional vector space, and also the finiteness (as a vector space) of $Ext^2_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}})$. This implies that we have, for any regular (a,b)-module, the inclusions

$$E^* \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset E^*.$$

They imply, thanks to the fact that $(E^b)^*$ and $(E^\sharp)^*$ have simple poles,

$$(E^*)^\sharp \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset (E^*)^b.$$

But the inclusion $(E^*)^b \subset E^*$ gives

$$E = (E^*)^* \subset ((E^*)^b)^* \subset ((E^\sharp)^*)^* = E^\sharp$$

and the minimality of $E^\sharp$ gives $((E^*)^b)^* = E^\sharp$ because $((E^*)^b)^*$ has a simple pole and contains E. Dualizing again gives $(E^\sharp)^* \simeq (E^*)^b$. The last equality is obtained in a similar way from $E^* \subset (E^*)^\sharp$. ■

Lemme 3.3.8 *Let V be an $\tilde{\mathcal{A}}-$module of finite dimension over $\mathbb{C}$. Then we have $Ext^i_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}}) = 0$ for $i = 0, 1$ and $Ext^2_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}})$ is again a $\tilde{\mathcal{A}}-$module (via the $\theta-$structure of $\tilde{\mathcal{A}}$) which is a finite dimensional vector space. Moreover it has the same dimension than V and there is a canonical $\tilde{\mathcal{A}}-$module isomorphism*

$$Ext^2_{\tilde{\mathcal{A}}}(Ext^2_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}}), \tilde{\mathcal{A}}) \simeq V.$$

PROOF. We begin by proving the first assertion of the lemma for the special case $V_\lambda := \tilde{\mathcal{A}}\big/\tilde{\mathcal{A}}.(a-\lambda) + \tilde{\mathcal{A}}.b$ for any $\lambda \in \mathbb{C}$. Let us show that we have the free resolution

$$0 \to \tilde{\mathcal{A}} \overset{\alpha}{\to} \tilde{\mathcal{A}}^2 \overset{\beta}{\to} \tilde{\mathcal{A}} \to V_\lambda \to 0$$

where $\alpha(x) := (x.b, -x.(a-b-\lambda))$, $\beta(u,v) := u.(a-\lambda) + v.b$. The map α is clearly injective and $\beta(\alpha(x)) = x.(b.a - \lambda.b - (a-b-\lambda).b) = 0$. If we have $\beta(u,v) = 0$ then $u \in \tilde{\mathcal{A}}.b$; let $u = x.b$. Then we get

$$x.(a-b-\lambda).b + v.b = 0 \quad \text{and so} \quad v = -x.(a-b-\lambda).$$

This gives the exactness of our resolution.
Now the $Ext^i_{\tilde{\mathcal{A}}}(V_\lambda, \tilde{\mathcal{A}})$ are given by the cohomology of the complex

$$0 \to \tilde{\mathcal{A}} \overset{\beta^*}{\to} \tilde{\mathcal{A}}^2 \overset{\alpha^*}{\to} \tilde{\mathcal{A}} \to 0.$$

The map $\beta^*(x) = ((a-\lambda).x, b.x)$ and $\alpha^*(u,v) = b.u - (a-b-\lambda).v$ are $\tilde{\mathcal{A}}-$linear for the $\theta-$structure of $\tilde{\mathcal{A}}$. Clearly β^* is injective and $\alpha^*(\beta^*(x)) \equiv 0$. If $\alpha^*(u,v) = 0$ set $v = b.y$ and conclude that $u = (a-\lambda).y$. This gives the vanishing of the Ext^i for $i = 0, 1$. The Ext^2 is the cokernel of β^* which is easily seen to be isomorphic to V_λ.
Consider now any finite dimensional $\tilde{\mathcal{A}}-$module V over $\mathbb{C}$. We make an induction on $\dim_{\mathbb{C}}(V)$ to prove the vanishing of the Ext^i for $i = 0, 1$ and the assertion on the dimension of the Ext^2.
The $\dim V = 1$ case is clear because reduced to the case $V = V_\lambda$ for some $\lambda \in \mathbb{C}$. Assume that the case $\dim V = p$ is proved, for $p \geq 1$ and consider some V with $\dim V = p+1$. Then $Ker\, b$ is not $\{0\}$ and is stable by a. Let $\lambda \in \mathbb{C}$ an eigenvalue of a acting on $Ker\, b$. Then a eigenvector generates in V a sub-$\tilde{\mathcal{A}}-$module isomorphic to V_λ.
The exact sequence of $\tilde{\mathcal{A}}-$modules

$$0 \to V_\lambda \to V \to W \to 0$$

where $W := V\big/V_\lambda$ has dimension p allows us to conclude, looking at the long exact sequence of Ext .
The last assertion follows from the remark that we produce a free resolution of $Ext^2_{\tilde{\mathcal{A}}}(V, \tilde{\mathcal{A}})$ by taking $Hom_{\tilde{\mathcal{A}}}(-, \tilde{\mathcal{A}})$ of a free (length two, see [B.97]) resolution of V because of the already proved vanishing of the Ext^i for $i = 0, 1$. Doing this again gives back the initial resolution (remark that we use here that the $\theta \circ \theta-$structure on $Hom_{\tilde{\mathcal{A}}}(Hom_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}), \tilde{\mathcal{A}})$ is the usual left structure on $\tilde{\mathcal{A}}$). ■

Corollaire 3.3.9 *For a simple pole (a,b) module E denote by $S(E)$ the spectrum of $b^{-1}.a$ acting on $E/b.E$. Then we have*

$$S(E^*) = -S(E).$$

PROOF. We make an induction on the rank of E. In rank 1 the result is clear because we have $E \simeq E_\lambda$ for some $\lambda \in \mathbb{C}$, and $S(E_\lambda) = \{\lambda\}$. But we know that $E_\lambda^* = E_{-\lambda}$.
Assume the assertion proved for any rank $p \geq 1$ simple pole (a,b)-module, and consider E with rank $p+1$. Using theorem 3.1.7, there exists $\lambda \in \mathbb{C}$ and an exact sequence (a,b)-modules

$$0 \to E_\lambda \to E \to F \to 0$$

where $rank(F) = p$ and where F has a simple pole (because a quotient of a simple pole (a,b)-module has a simple pole !). The exact sequence of vector spaces

$$0 \to E_\lambda/b.E_\lambda \to E/b.E \to F/b.F \to 0$$

shows that $S(E) = S(F) \cup \{\lambda\}$. Now proposition 3.3.7 gives the exact sequence

$$0 \to F^* \to E^* \to E_{-\lambda} \to 0$$

which implies, as before, $S(E^*) = S(F^*) \cup \{-\lambda\}$. The induction hypothesis $S(F^*) = -S(F)$ allows to conclude. ■

Lemme 3.3.10 *For any pair of (a,b)-modules E and F there is a canonical isomorphism of vector spaces*

$$D : Ext^1_{\tilde{\mathcal{A}}}(E,F) \to Ext^1_{\tilde{\mathcal{A}}}(F^*,E^*)$$

associated to the correspondance between 1-extensions (i.e. short exact sequences)

$$(0 \to F \to G \to E \to 0) \xrightarrow{D} (0 \to E^* \to G^* \to F^* \to 0).$$

PROOF. We have a obvious isomorphism of $\mathbb{C}[[b]]-$modules[3]

$$I : Hom_b(E,F) \to Hom_b(Hom_b(F,E_0), Hom_b(E,E_0)) \simeq Hom_b(F^*,E^*)$$

because $E_0 \simeq \mathbb{C}[[b]]$ as a $\mathbb{C}[[b]]-$module. But recall that $Ext^1_{\tilde{\mathcal{A}}}(E,F)$ (resp. $Ext^1_{\tilde{\mathcal{A}}}(F^*,E^*)$) is the cokernel of the $\mathbb{C}-$linear map "a" defined on $Hom_b(E,F)$ by the formula

$$(a.\varphi)(x) = \varphi(a.x) - a.\varphi(x)$$

So it is enough to check that the isomorphism I commutes with "a" in order to get an isomorphism between the cokernels of "a" in these two spaces.
Let $\varphi \in Hom_b(E,F)$ and $\xi \in F^*$. Then $I(\varphi)(\xi) = \varphi \circ \xi$. So, for $x \in E$ we have (using Λ to avoid too many "a")

$$\begin{aligned}
&\Lambda(I(\varphi)(\xi) = I(\varphi)(a.\xi) - a.(I(\varphi)(\xi)) \\
&\Lambda(I(\varphi)(\xi)(x) = (\varphi \circ \xi)(a.x) - a.\xi(\varphi(x)) - \big(\xi(\varphi(a.x)) - a.\xi(\varphi(x))\big) \\
&\qquad = \big[(\Lambda(\varphi)) \circ \xi\big](x) = I(\Lambda(\varphi))(x).
\end{aligned}$$

[3]but be carefull with the $b \to \check{b}$!

So $\Lambda \circ I = I \circ \Lambda$. The map I gives an isomorphism of complexes

$$\begin{array}{ccc} Hom_{a,b}(E,F) & \xrightarrow{\Lambda} & Hom_{a,b}(E,F) \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\ Hom_{a,b}(F^*,E^*) & \xrightarrow{\Lambda} & Hom_{a,b}(F^*,E^*) \end{array}$$

and this conclude the proof, using lemma 3.3.4. ■

For an (a,b)-module E and an integer $m \in \mathbb{N}$ it is clear that $b^m.E$ is again an (a,b)-module. This can be generalize for any $m \in \mathbb{C}$.

Définition 3.3.11 *For any (a,b)-module E and any complex number $m \in \mathbb{C}$ define the (a,b)-module $b^m.E$ as follows : as an $\mathbb{C}[[b]]-$module we let $b^m.E \simeq E \simeq \mathbb{C}[[b]]^{rank(E)}$; the operator a is defined as $a + m.b$.*

Precisely, this means that if $(e_1, \cdots, e_k)$ is a $\mathbb{C}[[b]]-$basis of E such that we have $a.e = M(b).e$ where $M \in End(\mathbb{C}^p) \otimes_{\mathbb{C}} \mathbb{C}[[b]]$, the (a,b)-module $b^m.E$ admit a basis, denote by $(b^m.e_1, \cdots, b^m.e_k)$, such that the operator a is defined by the relation $a.(b^m.e) := (M(b) + m.b.Id_k).(b^m.e)$.
Remark that for $m \in \mathbb{N}$ this notation is compatible with the preexisting one, because of the relation $a.b^m = b^m.(a + m.b)$.
For any $m \in \mathbb{N}$ there exists a canonical (a,b)-morphism

$$b^m.E \to E$$

which is an isomorphism of $b^m.E$ on $Im(b^m : E \to E)$. But remark that the map $b^m : E \to E$ is not $a-$linear (but the image is stable by a).
For any $m \in \mathbb{N}$ there is also a canonical (a,b)-morphism

$$E \to b^{-m}.E$$

which induces an isomorphism of E on $Im(b^m : b^{-m}.E \to b^{-m}.E)$. So we may write, via this canonical identification, $b^m.(b^{-m}.E) = E$.
It is easy to see that for any $m, m' \in \mathbb{C}$ we have a natural isomorphism

$$b^{m'}.(b^m.E) \simeq b^{m+m'}.E \quad \text{and also} \quad b^0.E \simeq E.$$

Remark. It is easy to show that for any $m \in \mathbb{C}$ there exists an unique $\mathbb{C}-$algebra automorphism

$$\eta_m : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \quad \text{such that} \quad \eta(1) = 1, \eta(b) = b \quad \text{and} \quad \eta(a) = a + m.b.$$

Using this automorphism, one can define a left $\tilde{\mathcal{A}}-$module $b^m.F$ for any left $\tilde{\mathcal{A}}-$module F and any $m \in \mathbb{C}$. This is, of course compatible with our definition in the context of (a,b)-modules. □

The behaviour of the correspondance $E \to b^m.E$ by duality is given by the following easy lemma; the proof is left as an exercice.

Lemme 3.3.12 *For any (a,b)-module E and any $m \in \mathbb{C}$ there is natural (a,b)-isomorphism*

$$(b^m.E)^* \to b^{-m}.E^*.$$

The following corollary of the lemma 3.2.3 and the proposition 3.3.7 allows to show that duality preserves the index.

Lemme 3.3.13 *Let E be a regular (a,b)-module. Then we have $\delta(E^*) = \delta(E)$.*

Proof. By definition $\delta(E)$ is the smallest integer $k \in \mathbb{N}$ such that $E^\sharp \subset b^{-k}.E$. Now $E^\sharp \subset b^{-m}.E$ implies by duality that $b^m.E^* \subset (E^*)^b$. So, by lemma 3.2.3, we have $m \geq \delta(E^*)$. This proves that $\delta(E) \leq \delta(E^*)$ and we obtain the equality by symetry. ■

Remark. Duality does not preserve the order of regularity : in the example given before the definition 3.2.2 we have $or(E) = 2$ and $or(E^*) = 1$. □

Let us conclude this section by an easy exercice.

Exercice. For any (a,b)-modules E, F and any $\lambda \in \mathbb{C}$ there are natural (a,b)-isomorphisms

1. $b^\lambda.E_\mu \simeq E_{\lambda+\mu}$.
2. $b^\lambda.Hom_{a,b}(E,F) \simeq Hom_{a,b}(b^{-\lambda}.E, F) \simeq Hom_{a,b}(E, b^\lambda.F)$.
3. Then deduce from the previous isomorphisms that $Hom_{a,b}(E, E_\lambda) \simeq b^{-\lambda}.E^*$, and $Ext^1_{\tilde{\mathcal{A}}}(E, E_\lambda) \simeq E^*/(a+\lambda.b).E^*$.

3.4 Width of a regular (a,b)-module.

Notation. For a complex number λ we shall note by $\tilde{\lambda}$ is class in $\mathbb{C}/\mathbb{Z}$. We shall order elements in each class *modulo* $\mathbb{Z}$ by its natural order on real parts. □

Définition 3.4.1 *Let E be a regular (a,b)-module and let $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$. We define the following complex numbers :*

$$\tilde{\lambda}_{min}(E) := \inf\{\lambda \in \tilde{\lambda} / \exists \text{ a non zero morphism } \quad E_\lambda \to E\}$$
$$\tilde{\lambda}_{max}(E) = \sup\{\lambda \in \tilde{\lambda} / \exists \text{ a non zero morphism } \quad E \to E_\lambda\}$$
$$L_{\tilde{\lambda}}(E) = \tilde{\lambda}_{max}(E) - \tilde{\lambda}_{min}(E) \in \mathbb{Z}$$
$$L(E) = \sup\{\tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \ / \ L_{\tilde{\lambda}}(E)\}$$

with the following conventions :

$$\inf\{\emptyset\} = +\infty, \ \sup\{\emptyset\} = -\infty \quad \text{and}$$
$$-\infty - \lambda = -\infty \quad \forall \lambda \in]-\infty, +\infty]$$
$$+\infty - \lambda = +\infty \quad \forall \lambda \in [-\infty, +\infty[.$$

We shall call $L(E)$ **the width of** E.

Remarks.

1. A non zero morphism $E_\lambda \to E$ is necessarily injective. Either its image is a normal submodule in E or there exists an integer $k \geq 1$ and a morphism $E_{\lambda-k} \to E$ whose image is normal an contains the image of the previous one.
2. In a dual way, a non zero morphism $E \to E_\lambda$ has an image equal to $b^k.E_\lambda \simeq E_{\lambda+k}$, where $k \in \mathbb{N}$.
3. A non zero morphism $E_\lambda \to E_\mu$ implies that λ lies in $\mu + \mathbb{N}$. It is possible that for some E we have $\tilde{\lambda}_{max}(E) < \tilde{\lambda}_{min}(E)$. For instance this is the case for the rank 2 regular (a,b)-module $E_{\lambda,\mu}$ from 3.1.6. So the width of a regular but not simple pole (a,b)-module is not necessarily a non negative integer.
4. Let E and F be regular (a,b)-modules. If there is a surjective morphism $E \to F$ then for all $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have $\tilde{\lambda}_{max}(E) \geq \tilde{\lambda}_{max}(F)$.
   If there is an injective morphism $E' \to E$ then for all $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have $\tilde{\lambda}_{min}(E) \leq \tilde{\lambda}_{min}(E')$.
5. Every submodule of E isomorphic to E_λ is contained in E^b. So we have $\tilde{\lambda}_{min}(E) = \tilde{\lambda}_{min}(E^b)$, for every regular (a,b)-module E and every $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$.
6. In a dual way, every morphism $E \to E_\lambda$ extends uniquely to a morphism $E^\sharp \to E_\lambda$ with the same image. So for every regular (a,b)-module E and every $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$, we get $\tilde{\lambda}_{max}(E) = \tilde{\lambda}_{max}(E^\sharp)$. □

Lemme 3.4.2 *1. Let E a simple pole (a,b)-module and let $S(E)$ denotes the spectrum of the linear map $b^{-1}.a : E/b.E \to E/b.E$, we have*

$$\tilde{\lambda}_{min}(E) = \inf\{\lambda \in S(E) \cap \tilde{\lambda}\} \quad \text{and} \quad \tilde{\lambda}_{max}(E) = \sup\{\lambda \in S(E) \cap \tilde{\lambda}\} \qquad (@)$$

2. For any regular (a,b)-module E we have

$$\widetilde{(-\lambda)}_{max}(E^*) = -\tilde{\lambda}_{min}(E) \qquad \widetilde{(-\lambda)}_{min}(E^*) = -\tilde{\lambda}_{max}(E).$$

This implies $L_{-\tilde{\lambda}}(E^*) = L_{\tilde{\lambda}}(E) \ \forall \tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$, *and so* $L(E^*) = L(E)$.

3. For any regular (a,b)-module E and any $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have equivalence between

$$\tilde{\lambda}_{min}(E) \neq +\infty \quad \text{and} \quad \tilde{\lambda}_{max}(E) \neq -\infty.$$

PROOF. Let E be a simple pole (a,b)-module. We have already seen (in proposition 3.1.4) that if $\lambda \in S(E)$ is minimal in its class modulo 1, there exists a non zero $x \in E$ such that $a.x = \lambda.b.x$. This implies that $\tilde{\lambda}_{min} \leq \inf\{\lambda \in S(E) \cap \tilde{\lambda}\}$. But the opposite inequality is obvious, so the first part of (@) is proved.
Using corollary 3.3.9 and the result already obtained for E^* gives

$$\widetilde{(-\lambda)}_{min}(E^*) = \inf\{-\lambda \in S(E^*) \cap \widetilde{(-\lambda)}\} = -\sup\{\lambda \in S(E) \cap \tilde{\lambda}\}.$$

So for $\mu = \sup\{\lambda \in S(E) \cap \tilde{\lambda}\}$ we have an exact sequence of (a,b)-modules

$$0 \to E_{-\mu} \to E^* \to F \to 0$$

and by duality, a surjective map $E \to E_\mu$. This implies $\tilde{\lambda}_{max} \geq \mu$. As, again, the opposite inequality is obvious, the second part of (@) is proved.
Let us prove now the relations in 2.
Remark first that these equalities are true for a simple pole (a,b)-module because of (@) and corollary 3.3.9.
For any regular (a,b)-module E we know that

$$\tilde{\lambda}_{min}(E) = \tilde{\lambda}_{min}(E^b) = \inf\{\lambda \in S(E^b) \cap \tilde{\lambda}\} \quad \text{and} \quad \widetilde{(-\lambda)}_{max}(E^*) = \widetilde{(-\lambda)}_{max}((E^*)^\sharp).$$

But we have

$$\widetilde{(-\lambda)}_{max}((E^*)^\sharp) = \sup\{-\lambda \in S((E^*)^\sharp) \cap \widetilde{(-\lambda)}\} = -\inf\{\lambda \in S(((E^*)^\sharp)^* \cap \tilde{\lambda}\}$$

because $(E^*)^\sharp$ has a simple pole, using corollary 3.3.9. So we obtain

$$\widetilde{(-\lambda)}_{max}(E^*) = -\tilde{\lambda}_{min}(E^b) = -\tilde{\lambda}_{min}(E)$$

because $(E^*)^\sharp)^* = E^b$ (see proposition 3.3.7).
The second relation is analoguous.
The equivalence in 3 is obvious in the simple pole case using (@).
The general case is an easy consequence using $E^b, E^\sharp$: if $\tilde{\lambda}_{min}(E) \neq +\infty$ so is $\tilde{\lambda}_{min}(E^\sharp)$ because $E \subset E^\sharp$. Then $\tilde{\lambda}_{max}(E^\sharp) \neq -\infty$ and so is $\tilde{\lambda}_{max}(E)$. The converse is analoguous using E^b. ■

REMARKS.

1. If E has a simple pole, we have $L_{\tilde{\lambda}}(E) \geq 0$ or $L_{\tilde{\lambda}}(E) = -\infty$ for any $\tilde{\lambda}$ in $\mathbb{C}/\mathbb{Z}$. So $L(E)$ is always ≥ 0.

2. In cases 1 and 2 of the proposition 3.1.6 the formula (@) gives the values of $\tilde{\lambda}_{min}$ and $\tilde{\lambda}_{max}$ for any $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$.
   For the remaining cases we can compute these numbers using the fact that we already know the corresponding E^b and $E^\sharp$ and the remark 5 and 6 before the preceeding lemma. □

Proposition 3.4.3 *Let E be a regular (a,b)-module and let $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$. Assume that $\lambda = \tilde{\lambda}_{min}(E) < +\infty$. Consider an exact sequence of (a,b)-modules*

$$0 \to E_\lambda \to E \xrightarrow{\pi} F \to 0.$$

Then we have for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$ the inequality

$$L_{\tilde{\mu}}(F) \leq L_{\tilde{\mu}}(E) + 1. \tag{i}$$

Proof. As $\tilde{\mu}_{max}(F) \leq \tilde{\mu}_{max}(E)$ for any $\mu \in \mathbb{C}$ it is enough to prove that we have $\tilde{\mu}_{min}(E) \leq \tilde{\mu}_{min}(F) + 1$ for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$.
Let begin by the case of $\tilde{\mu} = \tilde{\lambda}$. We want to show the inequality

$$\tilde{\lambda}_{min}(F) \geq \lambda - 1 \tag{ii}$$

Let $E_{\lambda-d} \hookrightarrow F$ with $d \geq 0$. The rank 2 (a,b)-module $G := \pi^{-1}(E_{\lambda-d})$ is contained in E, so $\lambda = \tilde{\lambda}_{min}(G)$. We have the exact sequence of (a,b)-modules

$$0 \to E_\lambda \to \pi^{-1}(E_{\lambda-d}) \xrightarrow{\pi} E_{\lambda-d} \to 0.$$

Now let us compare G with the list in proposition 3.1.6.
If G is in case 1, we have $E_{\lambda-d} \subset G$ so $d = 0$ because $\lambda = \tilde{\lambda}_{min}(G)$.
If G is in case 2, we have $\lambda - d = \lambda + n$ with $n \in \mathbb{N}$, so $d = 0$.
If G is in case 3, we have $G \simeq E_{\lambda,\lambda+k}$ with $k \in \mathbb{N}$. Then the theorem 3.1.7 gives $2\lambda - d = 2\lambda + k - 1$ and so $d = 1 - k \leq 1$.
If G is in case 4, we have $G \simeq E_{\lambda,\lambda+n}(\alpha)$. Again theorem 3.1.7 gives $2\lambda - d = 2\lambda + n - 1$ so $d = 1 - n \leq 0$ because $n \in \mathbb{N}^*$. So $d = 0$.
We conclude that we always have $d \leq 1$ and this proves (ii).

For $\tilde{\mu} \neq \tilde{\lambda}$ let us prove now the following inequality :

$$\tilde{\mu}_{min}(F) \leq \tilde{\mu}_{min}(E) \leq \tilde{\mu}_{min}(F) + 1. \tag{iii}$$

Consider an injective morphism $E_\mu \to E$ with $\mu = \tilde{\mu}_{min}(E)$. The restriction of π to E_μ is injective and so it gives $\tilde{\mu}_{min}(E) \geq \tilde{\mu}_{min}(F)$. Assume now that we have an injective morphism $E_{\mu'} \hookrightarrow F$ with $\mu' = \tilde{\mu}_{min}(F)$, and consider the rank 2 (a,b)-module $\pi^{-1}(E_{\mu'})$. Using the proposition 3.1.6 where only cases 1 or 3 are possible now, it can be easily check that (iii) is satisfied. ■

Remarks.

1. In the situation of the previous proposition we have either $\tilde{\lambda}_{min}(E) \geq \tilde{\lambda}_{max}(E)$ or $\tilde{\lambda}_{max}(E) = \tilde{\lambda}_{max}(F)$: Assume that we have $\lambda < \lambda' := \tilde{\lambda}_{max}(E)$. Then there exists a surjective morphism $q : E \to E_{\lambda'}$, and, as the restriction of q to E_λ is zero, the map q can be factorized and gives a surjective morphism $\tilde{q} : F \to E_{\lambda'}$. So we get $\tilde{\lambda}_{max}(E) \leq \tilde{\lambda}_{max}(F)$, and the desired equality thanks to the preceeding lemma.

2. We shall use later that in the situation of the previous proposition we have the inequality $\tilde{\lambda}_{max}(F) \leq \lambda + L(E)$. $\square$

Corollaire 3.4.4 *In the situation of the previous proposition we have the inequality $L(E) + rank(E) \geq L(F) + rank(F)$. So this integer is always positive for any non zero regular (a,b)-module.*

Proof. As the rank 1 case is obious, an easy induction on the rank of E using the propositions 3.1.4 and 3.4.3 gives the proof. $\blacksquare$

Examples.

1. The (a,b)-module
$$J_k(\lambda) := \tilde{\mathcal{A}}\big/\tilde{\mathcal{A}}.(a - (\lambda + k - 1).b)(a - (\lambda + k - 2).b) \cdots (a - \lambda.b)$$
which has rank k, satisfies $\lambda_{max} = \lambda$ and $\lambda_{min} = \lambda + k - 1$. So its width is $L(J_k(\lambda)) = -k + 1$.
To understand easily the (a,b)-module $J_k(\lambda)$ the reader may use the following alternative definition of it : there is a $\mathbb{C}[[b]]-$basis $(e_1, \cdots, e_k)$ in which the action of "a" is given by
$$a.e_1 = e_2 + \lambda.b.e_1, \quad a.e_2 = e_3 + (\lambda + 1).b.e_2, \cdots, a.e_k = (\lambda + k - 1).b.e_k.$$
2. The rank 2 (a,b)-module $E_\lambda \oplus E_{\lambda+n}$ has width n. This shows that, despite the fact that the width is always bigger than $-rank(E) + 1$, the width may be arbitrarily big, even for a rank 2 regular (a,b)-module. $\square$

4 Finite determination of regular (a,b)-modules.

4.1 Some more preliminaries.

Lemme 4.1.1 *Let E be a regular (a,b)-module of index $\delta(E) = k$. For $N \geq k+1$ the quotient map $q_N : E \to E\big/b^N.E$ induces a bijection between simple pole sub-(a,b)-modules F containing $b^k.E^\sharp$ and sub $\tilde{\mathcal{A}}-$modules $\mathcal{F} \subset E\big/b^N.E$ satisfying the following two conditions*

i) $a.\mathcal{F} \subset b.\mathcal{F}$;

ii) $b^k.E^\sharp\big/b^N.E \subset \mathcal{F}$.

Proof. It is clear that if F is a simple pole sub-(a,b)-module of E containing $b^k.E^\sharp$ the image $\mathcal{F} := q_N(F)$ is a $\tilde{\mathcal{A}}-$submodule of $E\big/b^N.E$ such that i) and ii) are fullfilled. Conversely, if a $\tilde{\mathcal{A}}-$submodule $\mathcal{F}$ satisfies i) and ii), let $F := q_N^{-1}(\mathcal{F})$. Of course, F is a sub-(a,b)-module of E and contains $b^k.E^\sharp$. The only point to see is that F has a simple pole. If $x \in F$ then $a.q_N(x) \in b.\mathcal{F}$ so $a.x \in b.F + b^N.E$. As $N \geq k+1$ we may write $a.x = b.y + b.z$ with $y \in F$ and $z \in b^{N-1}.E \subset b^k.E^\sharp \subset F$. This completes the proof. $\blacksquare$.

REMARKS.

1. we may replace $b^k.E^\sharp$ by $b^k.E$ in the second condition imposed on F and $\mathcal{F}$: if a simple pole (a,b)-submodule F contains $b^k.E$ it contains $b^k.E^\sharp$ by definition of $E^\sharp$. This allows to avoid the use of $E^\sharp$ in the previous lemma.

2. The biggest $\mathcal{F}$ satisfying i) and ii) corresponds to E^b. So we may recover E^b from the quotient $E\big/b^N.E$ for $N \geq \delta(E) + 1$. $\square$

Corollaire 4.1.2 *Let E be a regular (a,b)-module of order of regularity k. Fix $N \geq k+1$ and assume that we has an isomorphism of $\tilde{\mathcal{A}}-$modules*

$$\varphi : E\big/b^N.E \to E'\big/b^N.E'$$

where E' is an (a,b)-module. Then E' is regular and has order of regularity k. Moreover we have the equality $\varphi(E^b\big/b^N.E) = (E')^b\big/b^N.E'$.

PROOF. As k is the order of regularity of E we have $a^{k+1}.E \subset \sum_{j=0}^k a^j.b^{k-j+1}E$. The inequality $N \geq k+1$ gives $a^{k+1}.E\big/b^N.E \subset \sum_{j=0}^k a^j.b^{k-j+1}E\big/b^N.E$, and this is also true for $E'\big/b^N.E'$, and implies $a^{k+1}.E' \subset \sum_{j=0}^k a^j.b^{k-j+1}E'$. So the order of regularity of E' is at most k. We conclude that it is exactly k by symetry. The last stament comes from the second remark above, as $or(E) \geq \delta(E)$. $\blacksquare$

4.2 Finite determination for a rank one extension.

Lemme 4.2.1 *Let E be an (a,b)-module et fix a complex number λ. There exists $N(E,\lambda) \in \mathbb{N}$ such that for any $N \geq N(E,\lambda)$ we have the following inclusion :*

$$b^N.E \subset (a - \lambda.b).E.$$

PROOF. With the $b-$adic topology, E is a Frechet space. The $\mathbb{C}-$linear map $a - \lambda.b : E \to E$ is continuous. The finiteness theorem of [B.95], theorem 1.bis p.31 gives that the kernel and cokernel of this map are finite dimensional vector spaces. So the subspace $(a - \lambda.b).E$ is closed in E. This statement corresponds to the equality

$$\cap_{N\geq 0}\big[(a - \lambda.b).E + b^N.E\big] = (a - \lambda.b).E \qquad (@)$$

But the images of the subspaces $b^N.E$ in the finite dimensional vector space $E\big/(a-\lambda.b).E$ is a decreasing sequence. So it is stationnary, and, as the intersection is $\{0\}$ thanks to (@), the result follows. $\blacksquare$

Proposition 4.2.2 *Let F be an (a,b)-module and λ a complex number. Consider a short exact sequence of (a,b)-modules*

$$0 \to E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \qquad (@@)$$

where $E_\lambda := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda.b)$*. Then, for any* $N \geq N(F^*, -\lambda)$*, the extension* $(@@)$ *is uniquely determined by the following extension of* $\tilde{\mathcal{A}}-$*modules which are finite dimensional vectors spaces*

$$0 \to E_\lambda/b^N.E_\lambda \xrightarrow{\alpha} E/b^N.E \xrightarrow{\beta} F/b^N.F \to 0 \qquad (@@_N)$$

obtained from $(@@)$ *by "quotient by* b^N*".*

Comments. This statement needs some more explanations. Denote by K_N the kernel of the obvious map (forget "a")

$$ob_N : Ext^1_{\tilde{\mathcal{A}}}(F/b^N.F, E_\lambda/b^N.E_\lambda) \to Ext^1_b(F/b^N.F, E_\lambda/b^N.E_\lambda)$$

where $Ext^1_b(-,-)$ is a short notation for $Ext^1_{\mathbb{C}[[b]]}(-,-)$. The short exact sequence correspondance $(@@) \to (@@_N)$ gives a map

$$\delta_N : Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda) \to Ext^1_{\tilde{\mathcal{A}}}(F/b^N.F, E_\lambda/b^N.E_\lambda)$$

whose range lies in K_N, because the $\mathbb{C}[[b]]-$exact sequence $(@@)$ is split as F is $\mathbb{C}[[b]]-$free, and so is the exact sequence $(@@_N)$. The precise signification of the previous proposition is that for $N \geq N(F^*, -\lambda)$ the map δ_N is a $\mathbb{C}-$linear isomorphism between the vector spaces $Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda)$ and K_N. $\square$

Proof. As a first step to realize the map δ_N let us consider the following commutative diagramm of complex vector spaces, deduced from the exact sequences of $\tilde{\mathcal{A}}-$modules:

$$0 \to E_{\lambda+N} \to E_\lambda \to E_\lambda/b^N.E_\lambda \to 0$$
$$0 \to b^N.F \to F \to F/b^N.F \to 0$$

$$\begin{tikzcd}
Ext^1(F/b^N.F, E_{\lambda+N}) \arrow[r] \arrow[d] & Ext^1(F, E_{\lambda+N}) \arrow[r] \arrow[d, "\alpha"] & Ext^1(b^N.F, E_{\lambda+N}) \arrow[d] \\
Ext^1(F/b^N.F, E_\lambda) \arrow[r] \arrow[d] & Ext^1(F, E_\lambda) \arrow[r, "u"] \arrow[d, "\beta"] & Ext^1(b^N.F, E_\lambda) \arrow[d, "v"] \\
Ext^1(F/b^N.F, E_\lambda/b^N.E_\lambda) \arrow[r, "\gamma"] & Ext^1(F, E_\lambda/b^N.E_\lambda) \arrow[r, "w"] & Ext^1(b^N.F, E_\lambda/b^N.E_\lambda)
\end{tikzcd}$$

We have the following propreties :

1. The surjectivity of the map β is consequence of the vanishing of the vector space $Ext^2_{\tilde{\mathcal{A}}}(F, E_{\lambda+N})$ thanks to the proposition 3.3.7.

2. the vanishing of the composition $u \circ v$ is consequence of lemma 3.3.4 and of the fact that the restriction map

$$Hom_b(F, E_\lambda) \to Hom_b(b^N.F, E_\lambda) \to Hom_b(b^N.F, E_\lambda/b^N.E_\lambda)$$

is obviously zero.

3. So the map w is zero and γ is surjective.

4. The kernel of γ is given by the image of the injective map

$$\partial : Hom_{\tilde{\mathcal{A}}}(b^N.F, E_\lambda/b^N.E_\lambda) \hookrightarrow Ext^1_{\tilde{\mathcal{A}}}(F/b^N.F, E_\lambda/b^N.E_\lambda).$$

This is a consequence of the vanishing of the map

$$Ext^0_{\tilde{\mathcal{A}}}(F, E_\lambda/b^N.E_\lambda) \to Ext^0_{\tilde{\mathcal{A}}}(b^N.F, E_\lambda/b^N.E_\lambda).$$

Let us show now that for $N \geq N(F^*, -\lambda)$ the map α is zero. Using again the isomorphisms given by the lemma 3.3.4, α is induced by the obvious map $Hom_b(F, b^N.E_\lambda) \to Hom_b(F, E_\lambda)$, whose image is $b^N.Hom_b(F, E_\lambda)$. Denote respectively by G and H the (a,b)-modules given by $Hom_b(F, b^N.E_\lambda)$ and $Hom_b(F, E_\lambda)$ with the action of "a" defined by Λ (see 3.3.4). Then we have the following commutative diagramm

$$\begin{tikzcd}
G \arrow[r, "i"] \arrow[d] & H \arrow[d] \\
G/a.G \arrow[r] \arrow[d, "\simeq"] & H/a.H \arrow[d, "\simeq"] \\
Ext^1_{\tilde{\mathcal{A}}}(F, b^N.E_\lambda) \arrow[r, "\alpha"] & Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda)
\end{tikzcd}$$

and the image of i is $b^N.H$. So the map α will be zero as soon as $b^N.H \subset a.H$ and this is fullfilled for $N \geq N(H, 0) = N(F^*, -\lambda)$. This last equality coming from the isomorphisms

$$H/a.H \simeq Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda) \simeq Ext^1_{\tilde{\mathcal{A}}}(E_{-\lambda}, F^*) \simeq F^*/(a + \lambda.b).F^*$$

see the exercice concluding section 3.3.

Consider now the commutative diagramm

$$\begin{tikzcd}
 & 0 \arrow[d] & K_N \arrow[d, "i"] & Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda) \arrow[l, "\hat{\delta}_N"'] \arrow[dl, "\delta_N"'] \arrow[d, "\beta"] \\
0 \arrow[r] & Hom_{\tilde{\mathcal{A}}}(b^N.F, E_\lambda/b^N.E_\lambda) \arrow[r, "\partial"] \arrow[d, "ob_N"] & Ext^1_{\tilde{\mathcal{A}}}(F/b^N.F, E_\lambda/b^N.E_\lambda) \arrow[r, "\gamma"] \arrow[d, "ob_N"] & Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda/b^N.E_\lambda) \\
 & Hom_b(b^N.F, E_\lambda/b^N.E_\lambda) \arrow[r, "\simeq"] & Ext^1_b(F/b^N.F, E_\lambda/b^N.E_\lambda) &
\end{tikzcd}$$

The surjectivity of β implies that the map $i \circ \gamma$ is surjective (we know that the extensions in the image of δ_N comes from K_N, so δ_N factors in $\hat{\delta}_N \circ i$).
We have $i(K_N) \cap Im(\partial_N) = (0)$ because ob_N is injective on $Im(\partial_N)$.
So i induces an isomorphism of vector spaces from K_N to

$$Ext^1_{\tilde{\mathcal{A}}}(F/b^N.F, E_\lambda/b^N.E_\lambda)/Im(\partial_N) \overset{\gamma}{\simeq} Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda/b^N.E_\lambda) \overset{\beta^{-1}}{\simeq} Ext^1_{\tilde{\mathcal{A}}}(F, E_\lambda).$$

This completes the proof . ■

We shall need some bound for the integer $N(F^*, -\lambda)$ which appears in the previous proposition for the proof of our theorem.

Lemme 4.2.3 *Let G be a regular (a,b)-module and let $\mu \in \mathbb{C}$. A sufficient condition on $N \in \mathbb{N}$ in order to have the inclusion $b^N.G \subset (a - \mu.b).G$ is*

$$N \geq \mu - \tilde{\mu}_{min}(G) + \delta(G) + 2.$$

Proof. As we know that $\tilde{\mu}_{min}(G^b) = \tilde{\mu}_{min}(G)$, for $M \in \mathbb{N}$, the assumption $M > \mu - \tilde{\mu}_{min}(G)$ implies that $(a - (\mu - M).b).G^b = b.G^b$ (see the remark before proposition 3.1.4). By definition of the index of G we have $b^{\delta(G)}.G \subset G^b$. Combining both gives

$$b^{M+\delta(G)+1}.G \subset b^M.(a - (\mu - M).b).G = (a - \mu.b).b^M.G \subset (a - \mu.b).G.$$

Now let $N = M + \delta(G) + 1$; a sufficient condition on the integer N is now $N \geq \mu - \tilde{\mu}_{min}(G) + \delta(G) + 2$. ■

Corollaire 4.2.4 *A sufficient condition for $N \geq N(F^*, -\lambda)$ in the situation of prop. 4.2.2 in the regular case is that $N \geq or(E) + L(E) + rank(E) + 1$.*

Remark that the inequality $L(E) + rank(E) \geq 1$ for any non zero regular E implies that we have $or(E) + L(E) + rank(E) + 1 \geq or(E) + 2$.

Proof. We apply the previous lemma with $F^* = G$ and $\mu = -\lambda = -\tilde{\lambda}_{min}(E)$. The conclusion comes now from the following facts :

1. $-\widetilde{(-\lambda)}_{min}(F^*) = \tilde{\lambda}_{max}(F) \leq \lambda + L(E)$ this last inequality is proved in 3.4.3.
2. $\delta(F^*) = \delta(F) \leq or(F) \leq or(E)$ proved in 3.3.13 and 3.2.4 ■

4.3 The theorem.

Théorème 4.3.1 *Let E be a regular (a,b)-module. There exists an integer $N(E)$ such that for any (a,b)-module E', any integer $N \geq N(E)$ and any $\tilde{\mathcal{A}}-$isomorphism*

$$\varphi : E/b^N.E \to E'/b^N.E' \tag{1}$$

there exists an unique $\tilde{\mathcal{A}}-$isomorphism $\Phi : E \to E'$ inducing the given φ. Moreover the choice $N(E) = N_0(E) := or(E) + L(E) + rank(E) + 1$ is possible.

REMARKS.

1. It is easy to see that for a rank 1 regular (a,b)-module the integer 2 is the best possible.

2. In our final lemma 4.3.2 we show that the integer given in the theorem is optimal for the rank k (a,b)-module $J_k(\lambda)$, (defined in the lemma), for any $k \in \mathbb{N}^*$.

3. For the rank 2 (a,b)-modules $E_{\lambda,\mu}$ the integer given by the theorem is $or(E)+L(E)+2+1=3$ is again optimal, as it can be shown in the same maner that in our final lemma.

4. For the rank 2 simple pole (a,b)-module $E_\lambda(0)$ the integer given by the theorem is $3 = L(E) + rank(E) + 1$ and the best possible is 2 : the action of $b^{-1}.a$ on $E/b.E$ which is determined by $E/b^2.E$ characterizes this rank 2 regular (a,b)-module in the classification given in proposition 3.1.6.

5. For the (a,b)-module E associated to an holomorphic germ at the origine of $\mathbb{C}^{n+1}$ with an isolated singularity we have the uniform bounds $or(E) \leq n+1$ and $L(E) \leq n$ so the choice $N(E) = 2n + \mu + 2$ is always possible, where μ is the Milnor number (equal to the rank). □

PROOF. We shall make an induction on the rank of E. So we shall assume that the result is proved for a rank $p-1$ (a,b)-module and we shall consider a regular (a,b)-module E of rank $p \geq 1$, an (a,b)-module E', an integer $N \geq N_0(E)$ and an $\tilde{\mathcal{A}}-$isomorphism φ as in (1). From 4.1.2 we know that E' is then regular and has order of regularity $or(E') = or(E)$.
Choose now a complex number λ which is minimal *modulo* $\mathbb{Z}$ such there exists an exact sequence of (a,b)-module (so $\lambda = \tilde{\lambda}_{min}(E)$ with the terminology of §1.3)

$$0 \to E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0. \tag{2}$$

This exists from theorem 3.1.7. The (a,b)-module F has rank $p-1$ and from 4.2.4 we have $N_0(E) \geq N(F^*, -\lambda)$. So we know from 4.2.2 that the extension (2) is determined by the extension

$$0 \to E_\lambda/b^N.E_\lambda \xrightarrow{\alpha_N} E/b^N.E \xrightarrow{\beta_N} F/b^N.F \to 0. \tag{2_N}$$

Now, using the $\tilde{\mathcal{A}}-$isomorphism φ we obain an injective $\tilde{\mathcal{A}}-$linear map

$$j_N : E_\lambda/b^N.E_\lambda \hookrightarrow E'/b^N.E'.$$

Using the proposition 3.1.4 with the fact that $N \geq or(E')+2$ there exists a unique normal inclusion $j : E_\lambda \hookrightarrow E'$ inducing j_N.

Define $F' := E'/j(E_\lambda)$. Then F' is a rank $p-1$ (a,b)-module and the exact sequence

$$0 \to E_\lambda \xrightarrow{j} E' \to F' \to 0 \tag{2'}$$

induced the extension (2_N). Using the induction hypothesis, because the inequalities $or(E) \geq or(F)$ from 4.1.2 and $L(E) + rank(E) \geq L(F) + rank(F)$ from 3.4.4 implies $N_0(E) \geq N_0(F)$, we have a unique isomorphism $\Psi : F \to F'$ compatible with the one induced by φ between $F/b^N.F$ and $F'/b^N.F'$. Using 4.2.2, 4.2.4 and the inequality $N_0(E) \geq N(F^*, -\lambda)$ we have an unique isomorphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & E_\lambda \arrow[r, "\alpha"] \arrow[d, "="] & E \arrow[r, "\beta"] \arrow[d, "\Phi"] & F \arrow[r] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & E_\lambda \arrow[r, "j"] & E' \arrow[r] & F' \arrow[r] & 0
\end{tikzcd}$$

concluding the proof. ■

Lemme 4.3.2 *Let* $E := J_k(\lambda)$ *the rank* k *(a,b)-module defined by the* $\mathbb{C}[[b]]-$*basis* $e_1, \cdots, e_k$ *and by the following relations*

$$a.e_j = (\lambda + j - 1).b.e_j + e_{j+1} \quad \forall j \in [1, k]$$

with the convention $e_{k+1} = 0$. *We have* $\delta(E) = or(E) = k-1, L(E) = -k+1$. *The integer* $or(E) + L(E) + rank(E) + 1 = k + 1$ *is the best possible for the theorem.*

Proof. It is easy to see that the saturation $E^\sharp$ is generated by $e_1, b^{-1}.e_2, \cdots, b^{-k+1}.e_k$. This gives the equality $\delta(E) = or(E) = k - 1$.
Assume that we have an inclusion $E_\mu \hookrightarrow E$ such that $e_\mu \notin b.E$. Then there exists $(\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \setminus \{0\}$ such that

$$a.(\sum_{h=1}^{k} \alpha_h.e_h) = \mu.b(\sum_{h=1}^{k} \alpha_h.e_h) + b^2.E.$$

Then we obtain

$$\sum_{h=1}^{k} \alpha_h.\big((\lambda + j - 1).b.e_h + e_{h+1}\big) = \sum_{h=1}^{k} \alpha_h.\mu.b.e_h + b^2.E$$

and so $\alpha_1 = \cdots = \alpha_{k-1} = 0$ and we conclude that $\mu = \lambda + k - 1$.
An easy computation shows that $J_k(\lambda)^* = J_k(-\lambda-2k+2)$ and so we have $\lambda_{max} = \lambda$.
So $L(E) = -k + 1$.
Now we shall prove that the integer $k + 1$ is the best possible in the theorem 4.3.1 for $E = J_k(\lambda)$ by giving a regular (a,b)-module F such that $F/b^k.F \simeq E/b^k.E$ and not isomorphic to E.

Let consider the rank k (a,b)-module F defined by $\sum_{j=1}^k \mathbb{C}[[b]].e_j$ with the following relations

$$a.e_j = (\lambda + j - 1).b.e_j + e_{j+1} \quad \forall j \in [1, k]$$
$$a.e_k = (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h$$

Then define, for $\beta_1, \cdots, \beta_{k-1} \in \mathbb{C}$,

$$\varepsilon := e_k + \sum_{j=1}^{k-1} \beta_j.b^{k-j}.e_j.$$

We have

$$a.\varepsilon :=(\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h +$$
$$\sum_{j=1}^{k-1} \beta_j.\big[b^{k-j}.\big((\lambda + j - 1).b.e_j + e_{j+1}\big) + (k - j).b^{k-j+1}.e_j\big]$$
$$a.\varepsilon :=(\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \big(\alpha_h + \beta_h.(\lambda + k - 1) + \beta_{h-1}\big).b^{k-h+1}.e_h$$

Let now choose $\beta_1, \cdots, \beta_{k-1}$ such that we have

$$\alpha_h + \beta_h.(\lambda + k - 1) + \beta_{h-1} = (\lambda + k - 1 + \beta_{k-1}).\beta_h \quad \forall h \in [1, k - 1]$$

with the convention $\beta_0 = 0$. We obtain the system of equations

$$\alpha_h + \beta_{h-1} = \beta_{k-1}.\beta_h \quad \forall h \in [1, k - 1].$$

This implies, assuming $\beta_{k-1} \neq 0$, that β_{k-1} is solution of the equation

$$x^k = \alpha_{k-1}.x^{k-2} + \cdots + \alpha_2.x + \alpha_1.$$

Now choose $\alpha_2 = \cdots = \alpha_{k-1} = 0$ and $\alpha_1 := \rho^k$ with $\rho \in]0, 1[$. Then choose $\beta_j = \rho^{k-j} \quad \forall j \in [1, k-1]$. It is clear that the corresponding F_ρ satisfies $F/b^k.F \simeq E/b^k.E$ as $a.e_k = e_k + \rho.b^k.e_1$ in F_ρ. But the relation $a.\varepsilon = (\lambda + k - 1 + \rho^k).b.\varepsilon$ with $\varepsilon \neq 0$ shows that F_ρ cannot be isomorphic to $J_k(\lambda)$. ■

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